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Zassenhaus conjecture for A_5

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Abstract. We develop a criterion for rational conjugacy of torsion units of the integral group ring $\mathbb{Z}G$ of a finite group G, as also a necessary condition for an element of $\mathbb{Z}G$ to be a torsion unit, and apply them to verify the Zassenhaus conjecture in case when $G = A_5$.

Keywords. Zassenhaus conjecture; rational conjugacy; torsion unit.

1. A criterion for rational conjugacy

Let U be a complex $n \times n$ matrix with $U^k = 1$, $k \geqslant 1$. Let Z be a primitive kth root of unity, and let μ_l be the multiplicity of Z^l as an eigenvalue of U. Then

$$\mu_l = \frac{1}{k}\sum_{r=0}^{k-1} \mathrm{Tr}(U^r)\mathrm{Z}^{-lr}; \quad l = 0, 1, \ldots, k-1. \tag{1}$$

In particular, the numbers on the right hand side of (1) are non-negative integers with sum n.

This follows at once on noting that

$$\mathrm{Tr}(U^r) = \mu_0 1 + \mu_1 \mathrm{Z}^r + \cdots + \mu_{k-1}\mathrm{Z}^{(k-1)r}.$$

Let G be a finite group. Two torsion units u and v of $\mathbb{Z}G$ are rationally conjugate if and only if in each irreducible representation their matrices have the same characteristic polynomials. This is a consequence of Lemma 5 of [4] coupled with the fact that these matrices, being of finite order, are diagonalizable.

Let C be a conjugacy class in G. For an element $\alpha = \sum \alpha(g)g$ in $\mathbb{C}G$, we define its partial augmentation $\varepsilon_C(\alpha)$ over C by setting

$$\varepsilon_C(\alpha) = \sum_{g \in C} \alpha(g).$$

One checks immediately that $\varepsilon_C(\alpha\beta) = \varepsilon_C(\beta\alpha)$, and hence conjugate units in $\mathbb{C}G$ have the same partial augmentations.

Let u be a unit in $\mathbb{Z}G$, $u^k = 1$, $k \geqslant 1$. Let χ be any character of G of degree n, and let R be the corresponding representation. The multiplicity $\mu_l(u; \chi)$ of Z^l as an eigenvalue of $R(u)$ is given by

$$\mu_l(u;\chi)=\frac{1}{k}\sum_{r=0}^{k-1}\chi(u^r)Z^{-lr};\quad l=0,1,\ldots,k-1.$$

Collecting together those r which have the same g.c.d. with k we get

$$\mu_l(u;\chi)=\frac{1}{k}\sum_{d|k}\sum_{\substack{r \bmod k/d\\(r,k/d)=1}}\chi(u^{dr})Z^{-dlr}.$$

Since $(u^d)^{k/d}=1$, $\chi(u^d)$ is a sum of n (k/d)th roots $\varepsilon_1,\ldots,\varepsilon_n$ of unity; therefore for $(r,k/d)=1$

$$\chi(u^{dr})=\varepsilon_1^r+\cdots+\varepsilon_n^r=(\chi(u^d))^{\sigma_r},$$

where σ_r is the automorphism $Z^d\to Z^{dr}$ of $\mathbb{Q}(Z^d)$. It follows that

$$\mu_l(u;\chi)=\frac{1}{k}\sum_{d|k}\mathrm{Tr}_{\mathbb{Q}(Z^d)/\mathbb{Q}}(\chi(u^d)Z^{-dl});\quad l=0,1,\ldots,k-1. \tag{2}$$

In particular, we have:

Theorem 1. *Suppose that u is an element of $\mathbb{Z}G$ satisfying $u^k=1$, $k\geqslant 1$. Let Z be a primitive* kth *root of unity. Then for every integer l and every character χ of G, the number*

$$\frac{1}{k}\sum_{d|k}\mathrm{Tr}_{\mathbb{Q}(Z^d)/\mathbb{Q}}(\chi(u^d)Z^{-dl})$$

is a non-negative integer.

To obtain another consequence of (2), we notice that $\chi(u^d)$ depends only on the numbers $\varepsilon_C(u^d)$ and not on u as such. Therefore we obtain the following criterion for rational conjugacy of torsion units of $\mathbb{Z}G$.

Theorem 2. *Let u and v be units in $\mathbb{Z}G$ with $u^k=1=v^k$, $k\geqslant 1$. Then u and v are rationally conjugate if and only if $\varepsilon_C(u^d)=\varepsilon_C(v^d)$ for every divisor d of k and every conjugacy class C of G.*

In particular, for units of prime order we have

COROLLARY 1

Two units u and v in $\mathbb{Z}G$ satisfying $u^p=1=v^p$, p a prime, are rationally conjugate if and only if they have the same partial augmentations.

These results enable us to check the Zassenhaus conjecture in A_5.

2. Torsion units in A_5

Theorem 3. *Every normalized torsion unit in $\mathbb{Z}A_5$ is rationally conjugate to a group element.*

Proof. We denote by C_1, C_2, C_3, C_4 and C_5 the conjugacy classes in A_5 of 1, $S=(1\ 2)(3\ 4)$, $T=(1\ 2\ 3)$, $V=(1\ 2\ 3\ 4\ 5)$ and $V^2=(1\ 3\ 5\ 2\ 4)$ respectively. The

character table for A_5 is reproduced below [2, p. 319]:

	χ_1	χ_2	χ_3'	χ_4	χ_5
C_1	1	3	3	4	5
C_2	1	-1	-1	0	1
C_3	1	0	0	1	-1
C_4	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	-1	0
C_5	1	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$	-1	0

Let $u=\sum u(g)g$ be a normalized torsion unit in $\mathbb{Z}A_5$ of order $k>1$, and let

$$\nu_i=\varepsilon_{C_i}(u), \quad i=1,2,\ldots,5$$

be the partial augmentations of u. By Corollary 1.3 on page 45 of [5], we have

$$\nu_1=0; \tag{3}$$

since u is of augmentation 1,

$$\nu_2+\nu_3+\nu_4+\nu_5=1. \tag{3'}$$

For any character χ of A_5 of degree n,

$$\chi(u)=\nu_2\chi(S)+\nu_3\chi(T)+\nu_4\chi(V)+\nu_5\chi(V^2). \tag{4}$$

The possible values of k are divisors of 60 [1]. To prove the above theorem we need to show that a normalized unit of order 2, 3 or 5 is rationally conjugate to a group element, and that there is no normalized unit of order 4, 6, 10 or 15.

By Theorem 2.7 of [3] we have

$$\left\{\begin{array}{ll} \nu_3=\nu_4=\nu_5=0,\ \nu_2=1, & \text{when } k=2 \text{ or } 4 \\ \nu_2=\nu_4=\nu_5=0,\ \nu_3=1, & \text{when } k=3 \\ \nu_2=\nu_3=0 & \text{when } k=5 \\ \nu_4=\nu_5=0 & \text{when } k=6 \\ \nu_3=0 & \text{when } k=10 \\ \nu_2=0 & \text{when } k=15. \end{array}\right\} \tag{5}$$

When $k=2$ or 3, the partial augmentations of u are the same as those of S or T respectively; hence by Corollary 1, u is rationally conjugate to S or T. When $k=5$, we have by (2), with $Z=\exp(2\pi i/5)$, $K=\mathbb{Q}(Z)$

$$\begin{aligned}\mu_l(u;\chi_3)&=\tfrac{1}{5}[3+\mathrm{Tr}_{K/\mathbb{Q}}(Z^{-l}(\nu_4\chi_3(V)+\nu_5\chi_3(V^2)))]\\ &=\begin{cases}\nu_4 & \text{if } l=1\\ \nu_5 & \text{if } l=2.\end{cases}\end{aligned}$$

As the integers $\mu_l(u, \chi_3)$ are non-negative, the same is true of v_4 and v_5; since $v_4 + v_5 = 1$, we have

$$v_4 = 1, \quad v_5 = 0 \quad \text{or} \quad v_4 = 0, \quad v_5 = 1.$$

Thus the partial augmentations of u are the same as those of V or V^2; hence u is rationally conjugate either to V or V^2.

Since A_5 has no element of order 4, we see by Theorem 2.1 on page 177 of [5] that there are no normalized units of order 4 in $\mathbb{Z}A_5$. Thus it only remains to prove that k cannot be 6, 10 or 15.

Let $k = 6$, ω a primitive cube root of unity, and $\mathrm{Z} = -\omega$. We have, by the foregoing results, for any character χ of degree n,

$$\mu_l(u; \chi) = \tfrac{1}{6}[n + \chi(S)\mathrm{Z}^{-3l} + \mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\chi(T)\mathrm{Z}^{-2l} + \chi(u)\mathrm{Z}^{-l})].$$

It is clear from the character table of A_5 that $\chi(S)$, $\chi(T)$ and hence $\chi(u) = v_2\chi(S) + v_3\chi(T)$ are integers. Thus

$$\mu_l(u, \chi) = \tfrac{1}{6}[n + (-1)^l\chi(S) + \chi(T)\,\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}\,\omega^l + (-1)^l\chi(u)\,\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}\,\omega^{-l}]. \tag{6}$$

Taking $\chi = \chi_3$, we obtain

$$\mu_l(u; \chi_3) = \tfrac{1}{6}[3 + (-1)^{l+1} + (-1)^{l+1} v_2\,\mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}\,\omega^{-l}].$$

These being non-negative integers, we obtain on taking $l = 0$, 1 and 2 that $v_2 = -2$ and hence $v_3 = 3$. Now take $\chi = \chi_5$ in (6) to obtain

$$\mu_l(u; \chi_5) = \tfrac{1}{6}[5 + (-1)^l - \mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\omega^l + 5(-1)^l\omega^{-l})]$$

giving $\mu_0(u, \chi_5) = -1$ which is impossible. Thus there are no units of order 6.

Now let $k = 10$; then u^2 is rationally conjugate to either V or V^2. Replacing u by u^3, if necessary, we may assume that u^2 is rationally conjugate to V. Let $\zeta = \exp(2\pi i/5)$ and $\mathrm{Z} = -\zeta$. We have for a character χ of degree n

$$\mu_l(u, \chi) = \tfrac{1}{10}[n + (-1)^l\chi(S) + \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{3l}\chi(V)) + \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}((-1)^l\zeta^{4l}\chi(u))].$$

Taking $\chi = \chi_5$, we find that for every l

$$\tfrac{1}{10}[5 + (-1)^l + (-1)^l v_2\,\mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta^{4l})]$$

is a non-negative integer. One easily checks on taking $l = 0, 1$ and 5 that this is impossible.

Finally let $k = 15$; then u^3 is rationally conjugate to V or V^2. Replacing u by u^2, if necessary, we may assume that u^3 is conjugate to V. Let $\zeta = \exp(2\pi i/5)$, $\omega = \exp(2\pi i/3)$ and $\mathrm{Z} = \omega\zeta$.

We have, for a character χ of degree n

$$\mu_l(u; \chi) = \tfrac{1}{15}[n + \mathrm{Tr}_{\mathbb{Q}(\omega)/\mathbb{Q}}(\chi(T)\omega^l) + \mathrm{Tr}_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\chi(V)\zeta^{2l}) + \mathrm{Tr}_{\mathbb{Q}(\mathrm{Z})/\mathbb{Q}}(\chi(u)\mathrm{Z}^{-l})].$$

Taking $\chi = \chi_5$ and $l = 0$, 3 we see that

$$\frac{3 - 8v_3}{15} \quad \text{and} \quad \frac{3 + 2v_3}{15}$$

are non-negative integers. This being obviously impossible, we conclude that there is no normalized unit of order 15. This completes the proof of the theorem.

References

[1] Berman S D, On the equation $X^m = 1$ in an integral group ring, *Ukr. Math. Z.* **7** (1955) 253–261

[2] Burnside W, *Theory of groups of finite order* (Cambridge: Dover Publications) (1955)

[3] Marciniak Z, Ritter J, Sehgal S K and Weiss A, Torsion units in integral group rings of some metabelian groups, II. *J. Number Theory* **25** (1987) 340–352

[4] Polcino C Milies and Sehgal S K, Torsion units in integral group rings of metacyclic groups, *J. Number Theory* **19** (1984) 103–114

[5] Sehgal S K, *Topics in group rings* (New York: Dekker) (1978)

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Aperiodic rings, necklace rings and Witt vectors – II

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Abstract. In an earlier paper the author and Dr K Wehrhahn have introduced the concept of the aperiodic ring $\mathrm{Ap}(A)$ of a commutative ring A. It is a commutative ring equipped with two families of operators $V_r: \mathrm{Ap}(A) \to \mathrm{Ap}(A)$, $F_r: \mathrm{Ap}(A) \to \mathrm{Ap}(A)$ for every integer $r \geqslant 1$, called the Verschiebung and Frobenius operators. Let $D(A) = \{\sum_{k \geqslant 1} V_k \underline{S(a_k)} | a_k \in A\}$, where for any $a \in A$, $\underline{S(a)}$ is the element $S(a, 1), S(a, 2), S(a, 3), \ldots\ldots)$ of $\mathrm{Ap}(A)$. Let $\overline{\mathrm{W}(A)}$ denote the ring of Witt vectors of A. Let χ: $\mathrm{W}(A) \to \mathrm{Ap}(A)$ denote the map $\chi(a_1, a_2, a_3, \ldots\ldots) = \sum_{k \geqslant 1} V_k \underline{S(a_k)}$. We prove that χ is a ring homomorphism preserving the Verschiebung and Frobenius operators with image $\chi = D(A)$. Moreover χ: $\mathrm{W}(A) \to D(A)$ is an isomorphism if and only if the additive group of A is torsion-free.

Keywords. Necklace ring; aperiodic ring; Witt vectors; Verschiebung and Frobenius operators.

Introduction

In [2] Metropolis and Rota define the ring of Witt vectors $\mathrm{W}(A)$ of a commutative ring A by introducing an equivalence relation among strings via passage to unital power series. Then using the cyclotomic identity they try to establish a linkage between the necklace ring $\mathrm{Nr}(A)$ and the ring of Witt vectors $\mathrm{W}(A)$, whenever A is an integral domain of characteristic zero. However, as pointed out already in [3] their results are valid only when A is a commutative ring containing Q as a subring. For their results to be valid A need not be an integral domain. Let $\mathscr{A}$ denote the class of commutative rings A with the property that the additive group of A is torsion free. In [3] it was shown that for any $A \in \mathscr{A}$, $\mathrm{W}(A)$ is isomorphic to a suitably defined subring $\Delta(A)$ of $\mathrm{Nr}(A_0)$ where $A_0 = A \underset{Z}{\otimes} Q$ is the rationalization of A. For every commutative ring A, the concept of the aperiodic ring $\mathrm{Ap}(A)$ was introduced in [3]. While the definition of the necklace ring was motivated by the necklace polynomials and the identities between them, the concept of the aperiodic ring was derived from the cyclic polynomials and the identities between them. For any $A \in \mathscr{A}$, it was shown that $\mathrm{W}(A)$ is isomorphic to a suitably defined subring $\mathrm{D}(A)$ of $\mathrm{Ap}(A)$ in [3]. The proofs in [3] also depend on the cyclotomic identity. In our present paper we start with the more customary definition of the ring $\mathrm{W}(A)$ of Witt vectors, as defined on page 117 of §17 of [1]. For any $A \in \mathscr{A}$ we construct explicit isomorphisms between

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W(A) and the subring D(A) of Ap(A) (resp. the subring $\Delta(A)$ of Nr(A_0)) defined in [3]. The proofs in the present paper are purely algebraic and do not need the cyclotomic identity.

1. The rings Nr(A), Ap(A) and Gh(A) for any commutative ring A

Throughout this paper A will denote a commutative ring with $1 \neq 0$.

For any integer $n \geqslant 1$, let $S(\alpha, n)$ (resp. $M(\alpha, n)$) denote the number of aperiodic words (resp. primitive necklaces) of length n that can be formed out of an alphabet containing α letters, where α is an integer $\geqslant 0$. Then it is well-known and easy to see that

$$S(\alpha, n) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\alpha^d \tag{1}$$

and that

$$M(\alpha, n) = \frac{S(\alpha, n)}{n} = \frac{1}{n}\sum_{d|n} \mu\left(\frac{n}{d}\right)\alpha^d, \tag{2}$$

where μ is the classical Möebius function. The polynomials $S(X,n) \in Z[X]$ (resp. $M(X,n) = [S(X,n)/n] \in Q[X]$) are known as the cyclic (resp. necklace) polynomials. The following identities are proved in [3], for all integers $n \geqslant 1$, $r \geqslant 1$ and $s \geqslant 1$.

$$S(XY, n) = \sum_{[i,j]=n} S(X,i)S(Y,j) \tag{3}$$

$$S(X^r, n) = \sum_{\{j|[r,j]=rn\}} S(X,j) \tag{4}$$

$$S(X^{s/(r,s)}Y^{r/(r,s)}, n) = \sum_{\{i,j|ij(r,s)=n(ri,sj)\}} S(X,i)S(Y,j) \tag{5}$$

$$M(XY, n) = \sum_{[i,j]=n} (i,j)M(X,i)M(Y,j) \tag{6}$$

$$M(X^r, n) = \sum_{\{j|[r,j]=nr\}} \frac{j}{n} M(X,j) \tag{7}$$

$$(r,s)M(X^{s/(r,s)}Y^{r/(r,s)}, n) = \sum_{\{i,j|ij(r,s)=n(ri,sj)\}} (ri,sj)M(X,i)M(Y,j). \tag{8}$$

Here (3), (4), (5) are identities in $Z[X,Y]$, whereas (6), (7), (8) are identities in $Q[X,Y]$. As usual, for any two integers $k \geqslant 1$, $l \geqslant 1$, their g.c.d. is denoted by (k,l) and their l.c.m. is denoted by $[k,l]$.

The definition of the necklace ring Nr(A) is motivated by identity (6) and the definition of the Frobenius operators F_r: Nr(A) $\to$ Nr(A) is motivated by identity (7).

DEFINITION 1.1 [2]

The elements of Nr(A) are infinite sequences $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ with $a_k \in A$ for all $k \geqslant 1$.

If $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ and $\mathbf{b} = (b_1, b_2, b_3, \ldots)$ are any two elements of Nr(A) their sum $\mathbf{a} + \mathbf{b}$ and product $\mathbf{a} * \mathbf{b}$ in Nr(A) are given by $\mathbf{a} + \mathbf{b} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3, \ldots)$,

$\mathbf{a}*\mathbf{b}=\mathbf{c}\ (c_1, c_2, c_3, \ldots)$ where

$$c_n = \sum_{[i,j]=n} (i,j)a_i b_j. \tag{9}$$

Then Nr(A) is a commutative ring with $\mathbf{0}=(0,0,0,\ldots)$ as the zero element and $\mathbf{1}=(1,0,0,0,\ldots)$ as the identity element. Nr(A) is called the necklace ring of A.

The motivation for the definition of the aperiodic ring Ap(A) is identity (3). The definition of the operators F_r: Ap(A)$\to$Ap(A) is motivated by identity (4).

DEFINITION 1.2 [3]

The elements of Ap(A) are infinite sequences $\mathbf{a}=(a_1, a_2, a_3, \ldots)$ with $a_k \in A$ for all $k \geqslant 1$. If $\mathbf{a}=(a_1, a_2, a_3, \ldots)$ and $\mathbf{b}=(b_1, b_2, b_3, \ldots)$ are any two elements of Ap(A) their sum $\mathbf{a}+\mathbf{b}$ and product $\mathbf{a}\circ\mathbf{b}$ in Ap(A) are given by

$$\mathbf{a}+\mathbf{b}=(a_1+b_1, a_2+b_2, a_3+b_3, \ldots)$$

$$\mathbf{a}\circ\mathbf{b}=\mathbf{u}=(u_1, u_2, u_3, \ldots)$$

where

$$u_n = \sum_{[i,j]=n} a_i b_j. \tag{10}$$

Ap(A) is a commutative ring with $\mathbf{0}=(0,0,0,\ldots)$ as the zero element and $\mathbf{1}=(1,0,0,0,\ldots)$ as the identity element. Ap(A) will be referred to as the aperiodic ring of A.

DEFINITION 1.3

The Ghost ring of A, denoted by Gh(A) is by definition the direct product of a countable number of copies of A. Thus the elements of Gh(A) are infinite sequences $\mathbf{g}=(g_1, g_2, g_3, \ldots)$ with $g_k \in A$ for all $k \geqslant 1$. Addition and multiplication are defined coordinate-wise. If $\mathbf{g}=(g_1, g_2, g_3, \ldots)$ and $\mathbf{h}=(h_1, h_2, h_3, \ldots)$ are any two elements of Gh(A), then

$$\mathbf{g}+\mathbf{h}=(g_1+h_1, g_2+h_2, g_3+h_3, \ldots)$$

$$\mathbf{gh}=(g_1h_1, g_2h_2, g_3h_3, \ldots)$$

in Gh(A).

For each integer $r \geqslant 1$, there are additive homomorphisms V_r:Nr(A)$\to$Nr(A), V_r:Ap(A)$\to$Ap(A), V_r:Gh(A)$\to$Gh(A) called the Verschiebung operators and F_r:Nr(A)$\to$Nr(A), F_r:Ap(A)$\to$Ap(A), F_r:Gh(A)$\to$Gh(A) called the Frobenius operators.

For Nr(A), the operators V_r:Nr(A)$\to$Nr(A) and F_r:Nr(A)$\to$Nr(A) are defined as follows:

For any $\mathbf{a}=(a_1, a_2, a_3, \ldots\ldots)\in$Nr($A$),

$$V_r\mathbf{a}=\mathbf{x}=(x_1, x_2, x_3, \ldots) \text{ and}$$

$$F_r\mathbf{a}=\mathbf{y}=(y_1, y_2, y_3, \ldots)$$

where

$$x_n = \begin{cases} 0 & \text{if } r \nmid n \\ a_n & \text{if } r \mid n \end{cases} \tag{11}$$

and

$$y_n = \sum_{\{j \mid [r,j] = rn\}} \frac{j}{n} a_j. \tag{12}$$

In the case of Ap(A) the operators V_r:Ap(A) → Ap(A) and F_r:Ap(A) → Ap(A) are defined by the following formulae. For any $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in$ Ap(A),

$$V_r\mathbf{a} = \mathbf{v} = (v_1, v_2, v_3, \ldots) \text{ and}$$

$$F_r\mathbf{a} = \mathbf{w} = (w_1, w_2, w_3, \ldots)$$

where

$$v_n = \begin{cases} 0 & \text{if } r \nmid n \\ r a_{n/r} & \text{if } r \mid n \end{cases} \tag{13}$$

and

$$w_n = \sum_{\{j \mid [r,j] = nr\}.} a_j. \tag{14}$$

The operators V_r:Gh(A) → Gh(A) and F_r:Gh(A) → Gh(A) are defined as follows: For any $\mathbf{g} = (g_1, g_2, g_3, \ldots) \in$ Gh(A),

$$V_r\mathbf{g} = \mathbf{h} = (h_1, h_2, h_3, \ldots) \text{ and}$$

$$F_r\mathbf{g} = \mathbf{e} = (e_1, e_2, e_3, \ldots)$$

where

$$h_n = \begin{cases} 0 & \text{if } r \nmid n \\ r g_{n/r} & \text{if } r \mid n \end{cases} \tag{15}$$

and

$$e_n = g_{nr}. \tag{16}$$

Let ψ:Nr(A) → Ap(A), ϕ:Ap(A) → Gh(A) and θ:Nr(A) → Gh(A) be defined as follows: For any $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in$ Nr(A), $\psi(\mathbf{a}) = \mathbf{b} = (b_1, b_2, b_3, \ldots)$ in Ap(A) where

$$b_n = n a_n. \tag{17}$$

For any $\mathbf{c} = (c_1, c_2, c_3, \ldots) \in$ Ap(A), $\phi(\mathbf{c}) = \mathbf{g} = (g_1, g_2, g_3, \ldots)$ in Gh(A) where

$$g_n = \sum_{d \mid n} c_d. \tag{18}$$

For any $\mathbf{a}=(a_1,a_2,a_3,\ldots)\in\mathrm{Nr}(A)$, $\theta(\mathbf{a})=\mathbf{h}=(h_1,h_2,h_3,\ldots)$ in $\mathrm{Gh}(A)$ where

$$h_n=\sum_{d|n} da_d. \tag{19}$$

The following theorem summarizes some of the results proved in [3].

Theorem 1.4. (*i*) *For any commutative ring A, $\phi\colon \mathrm{Ap}(A)\to\mathrm{Gh}(A)$ is an isomorphism of rings; $\theta\colon\mathrm{Nr}(A)\to\mathrm{Gh}(A)$ and $\psi\colon\mathrm{Nr}(A)\to\mathrm{Ap}(A)$ are homomorphisms of rings. All the three maps θ, ψ and ϕ preserve the Verschiebung and Frobenius operators.*
(*ii*) *The following diagram is commutative*

$$\begin{tikzcd} \mathrm{Nr}(A) \arrow[rr, "\psi"] \arrow[dr, "\theta"'] & & \mathrm{Ap}(A) \arrow[dl, "\phi"] \\ & \mathrm{Gh}(A) & \end{tikzcd}$$

Diagram 1

(*iii*) *θ (hence ψ) is injective whenever the additive group of A is torsion-free. θ (hence ψ) is surjective whenever the additive group of A is divisible. In particular θ and ψ are isomorphisms whenever A contains Q as a subring.*

Refer to 4.1, 4.2, 5.1, 5.2 and 5.3 of [3]. Observe that the proofs of these results in [3] are purely algebraic and do not depend on the cyclotomic identity.

2. The ring W(A) of Witt vectors of a commutative ring A

In this section we briefly recall the customary definition of the ring $\mathrm{W}(A)$ of Witt vectors of a commutative ring A, as described on page 117, §7 of [1]. Let $\{X_n\}_{n\geqslant 1}$ be an infinite sequence of indeterminates over Z (the ring of integers). We denote a typical element of $Z[X_1,X_2,X_3,\ldots]$ by $p(X)$. Let $w_n(X)$ for $n\geqslant 1$ be the sequence of elements of $Z(X_1,X_2,X_3,\ldots]$ given by

$$w_n(X)=\sum_{d|n} dX_d^{n/d}.$$

Observe that $w_n(X)\in Z[X_1,\ldots,X_n]$. Let ϕ be a polynomial over Z in two indeterminates (one or both of which could be dummy). Then it is shown in §7, page 116 of [1] that there exists a unique sequence of elements $\varphi_k(X_1,\ldots,X_k;Y_1,\ldots,Y_k)\in Z[X_1,\ldots,X_k,Y_1,\ldots,Y_k]$, completely determined by ϕ and satisfying

$$\phi(w_n(X_1,\ldots,X_n),w_n(Y_1,\ldots,Y_n))=w_n(\varphi_1(X_1;Y_1),\varphi_2(X_1,X_2;Y_1,Y_2),\ldots). \tag{20}$$

In particular there exist sequences of polynomials

$$\Sigma_k(X_1,X_2,\ldots,X_k;Y_1,\ldots,Y_k)\in Z[X_1,\ldots,X_k,Y_1,\ldots,Y_k],$$

$$\Pi_k(X_1,X_2,\ldots,X_k;Y_1,\ldots,Y_k)\in Z[X_1,\ldots,X_k,Y_1,\ldots,Y_k] \text{ and}$$

$$\iota_k(X_1,\ldots,X_k)\in Z[X_1,\ldots,X_k] \text{ satisfying}$$

$$w_n(X) + w_n(Y) = w_n(\Sigma_1(X_1; Y_1), \Sigma_2(X_1, X_2; Y_1, Y_2), \ldots) \tag{21}$$

$$w_n(X)w_n(Y) = w_n(\pi_1(X_1; Y_1), \pi_2(X_1, X_2; Y_1, Y_2), \ldots) \tag{22}$$

and

$$-w_n(X) = w_n(\iota_1(X_1), \iota_2(X_1, X_2), \iota_3(X_1, X_2, X_3), \ldots) \tag{23}$$

DEFINITION 2.1 [1].

The elements of W(A) are infinite sequences $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ with $a_k \in A$ for all $k \geqslant 1$. If $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ and $\mathbf{b} = (b_1, b_2, b_3, \ldots)$ their sum $\mathbf{a} \underset{w}{+} \mathbf{b}$ and product $\mathbf{a} \underset{w}{\cdot} \mathbf{b}$ in W(A) are defined by

$$\mathbf{a} \underset{w}{+} \mathbf{b} = (\Sigma_1(a_1; b_1), \Sigma_2(a_1, a_2; b_1, b_2), \Sigma_3(a_1, a_2, a_3; b_1, b_2, b_3), \ldots) \tag{24}$$

$$\mathbf{a} \underset{w}{\cdot} \mathbf{b} = (\pi_1(a_1; b_1), \pi_2(a_1, a_2; b_1, b_2), \pi_3(a_1, a_2, a_3; b_1, b_2, b_3), \ldots) \tag{25}$$

It is known that W(A) is a commutative ring with $\mathbf{0} = (0, 0, 0, \ldots)$ as the zero element and $\mathbf{1} = (1, 0, 0, 0, \ldots)$ as the identity element. For any $\mathbf{a} = (a_1, a_2, a_3, \ldots\ldots)$ the element $(\iota_1(a_1), \iota_2(a_1, a_2), \iota_3(a_1, a_2, a_3), \ldots)$ is the additive inverse of $\mathbf{a}$ in W(A). Also W(A) comes equipped with Verschiebung and Frobenius operators V_r and F_r for each integer $r \geqslant 1$. Denote the set of unital power series over A by $1 + tA[[t]]$. It is well-known that the map $E\colon \mathrm{W}(A) \to 1 + tA[[t]]$ defined by

$$E(a_1, a_2, a_3, \ldots) = \prod_{k \geqslant 1} \left(\frac{1}{1 - a_k t^k} \right)$$

is a bijection [Proposition 17.2.9, page 121 of [1]]. Here the product occurring on the right hand side is the usual product of power series. Moreover for $\mathbf{a}$, $\mathbf{b}$ in W(A), it is known that $E(\mathbf{a} \underset{w}{+} \mathbf{b}) = E(\mathbf{a})E(\mathbf{b})$ where again $E(\mathbf{a})E(\mathbf{b})$ is the usual product of power series. To define V_r and F_r on W(A), one defines V_r and F_r on $1 + tA[[t]]$ and uses the bijection E to pull them back to W(A).

For any $a \in A$, and any integer $r \geqslant 1$ we set

$$V_r\left(\frac{1}{1 - at^k} \right) = \frac{1}{1 - at^{kr}} \tag{26}$$

$$F_r\left(\frac{1}{1 - at^k} \right) = \left(\frac{1}{1 - a^{r/(r,k)} t^{k/(r,k)}} \right)^{(r,k)}. \tag{27}$$

Thus in general V_r and F_r are defined on $1 + tA[[t]]$ by

$$V_r\left(\prod_{k \geqslant 1} \left(\frac{1}{1 - a_k t^k} \right) \right) = \prod_{k \geqslant 1} \left(\frac{1}{1 - a_k t^{kr}} \right) \tag{28}$$

$$F_r\left(\prod_{k \geqslant 1} \left(\frac{1}{1 - a_k t^k} \right) \right) = \prod_{k \geqslant 1} \left(\frac{1}{1 - a_k^{r/(r,k)} t^{k/(r,k)}} \right)^{(r,k)}. \tag{29}$$

For any $a \in A$, let $\underline{S(a)} = (S(a, 1), S(a, 2), S(a, 3), \ldots) \in \mathrm{Ap}(A)$. For any sequence $\{a_k\}_{k \geqslant 1}$ of elements of A, the element $\sum_{k \geqslant 1} V_r \underline{S(a_k)}$ is a well defined element of Ap(A). Let $\mathrm{D}(A) = \{\sum_{k \geqslant 1} V_k \underline{S(a_k)} \mid a_k \in A \text{ for all } k \geqslant 1\}$. The main result of the present paper could be stated as follows:

Theorem 2.2. *For any commutative ring A, $\mathrm{D}(A)$ is a subring of $\mathrm{Ap}(A)$ invariant under all the operators V_r and F_r for $r \geqslant 1$. The map $\chi: \mathrm{W}(A) \to \mathrm{D}(A)$ defined by*

$$\chi(a_1, a_2, a_3, \ldots) = \sum_{k \geqslant 1} V_k \underline{S(a_k)} \tag{30}$$

is an onto ring homomorphism preserving the Verschiebung and the Frobenius operators. Moreover χ is an isomorphism if and only if $A \in \mathscr{A}$.

Proof. Consider the composite map $\phi\chi: \mathrm{W}(A) \to \mathrm{Gh}(A)$ where $\phi: \mathrm{Ap}(A) \to \mathrm{Gh}(A)$ is the ring isomorphism defined by (18) in §1 of the present paper. For any $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in \mathrm{W}(A)$, we have

$$\phi\chi(\mathbf{a}) = \phi\left(\sum_{k \geqslant 1} V_k \underline{S(a_k)}\right)$$

$$= \mathbf{g} = (g_1, g_2, g_3, \ldots) \in \mathrm{Gh(A)},$$

where $g_n = \sum_{d|n} d a_d^{n/d}$, from proposition 5.9 of [3]

$$= w_n(a_1, \ldots, a_n). \tag{31}$$

Let $\mathbf{a}$ $(a_1, a_2, a_3, \ldots)$ and $\mathbf{b} = (b_1, b_2, b_3, \ldots)$ be any two elements of $\mathrm{W}(A)$. Then $\phi\chi(\mathbf{a} \underset{w}{+} \mathbf{b}) = w_n(\Sigma_1(a_1; b_1), \Sigma_2(a_1, a_2; b_1, b_2), \ldots)$ by (31)

$$= w_n(a_1, \ldots, a_n) + w_n(b_1, \ldots, b_n). \tag{32}$$

Similarly $\phi\chi(\mathbf{a} \underset{w}{\cdot} \mathbf{b}) = w_n(\pi_1(a_1; b_1), \pi_2(a_1, a_2; b_1 b_2), \ldots)$

$$= w_n(a_1, \ldots, a_n) w_n(b_1, \ldots, b_n). \tag{33}$$

If $\phi\chi(\mathbf{a}) = \mathbf{g} = (g_1, g_2, g_3, \ldots)$ and $\phi\chi(\mathbf{b}) = \mathbf{h} = (h_1, h_2, h_3, \ldots)$ then (31), (32) and (33) yield

$$\left.\begin{aligned} \phi\chi(\mathbf{a} \underset{w}{+} \mathbf{b}) &= \mathbf{g} + \mathbf{h} \\ \phi\chi(\mathbf{a} \underset{w}{\cdot} \mathbf{b}) &= \mathbf{g} \cdot \mathbf{h} \end{aligned}\right\} \text{in } \mathrm{Gh}(A). \text{ Clearly } w_n(1, 0, 0, \ldots, 0) = 1 \text{ for all}$$

$n \geqslant 1$, yielding $\phi\chi(1, 0, 0, 0, 0, 0, \ldots) = (1, 1, 1, 1, \ldots) = \mathbf{1} \in \mathrm{Gh}(A)$. Thus $\phi\chi: \mathrm{W}(A) \to \mathrm{Gh}(A)$ is a ring homomorphism. Since $\phi: \mathrm{Ap}(A) \to \mathrm{Gh}(A)$ is a ring isomorphism we see that $\chi: \mathrm{W}(A) \to \mathrm{Ap}(A)$ is a ring homomorphism. Clearly $\mathrm{D}(A) = \text{Image } \chi$. Hence $\mathrm{D}(A)$ is subring of $\mathrm{Ap}(A)$.

Next, we show that $\chi: \mathrm{W}(A) \to \mathrm{Ap}(A)$ preserves the operators V_r and F_r for $r \geqslant 1$. Observe that $\chi = HE$ where H: $1 + tA[[t]] \to \mathrm{Ap}(A)$ is defined by

$$H\left(\prod_{k \geqslant 1}\left(\frac{1}{1 - a_k t^k}\right)\right) = \sum_{k \geqslant 1} V_k \underline{S(a_k)} \cdot E: \mathrm{W}(A) \to 1 + tA[[t]]$$

is a bijection carrying addition in $\mathrm{W}(A)$ to usual multiplication of power series. Since $\chi: \mathrm{W}(A) \to \mathrm{Ap}(A)$ preserves addition, it follows that

$$H\left(\prod_{k \geqslant 1}\left(\frac{1}{1 - a_k t^k}\right)^{l_k}\right) = \sum_{k \geqslant 1} l_k V_k \underline{S(a_k)}$$

for any sequence of integers l_k. To pr·ve t‥t χ preserves V_r and F_r we have only to show that

$$H\left(V_r\left(\prod_{k\geq 1}\left(\frac{1}{1-a_kt^k}\right)\right)\right)=V_r\left(\sum_{k\geq 1}V_k\underline{S(a_k)}\right) \tag{34}$$

and

$$H\left(F_r\left(\prod_{k\geq 1}\left(\frac{1}{1-a_kt^k}\right)\right)\right)=F_r\left(\sum_{k\geq 1}V_k\underline{S(a_k)}\right). \tag{35}$$

Now

$$\begin{aligned}H\left(V_r\left(\prod_{k\geq 1}\left(\frac{1}{1-a_kt^k}\right)\right)\right)&=H\left(\prod_{k\geq 1}\left(\frac{1}{1-a_kt^{kr}}\right)\right)\\&=\sum_{k\geq 1}V_{kr}\underline{S(a_k)}\\&=\sum_{k\geq 1}V_rV_k\underline{S(a_k)}\\&=V_r\left[\sum_{k\geq 1}V_k\underline{S(a_k)}\right],\end{aligned}$$

yielding (34).
Also

$$\begin{aligned}H\left(F_r\left(\prod_{k\geq 1}\left(\frac{1}{1-a_kt^k}\right)\right)\right)&=H\left(\prod_{k\geq 1}\left(\frac{1}{1-a_k^{r/(r,k)}t^{k/(r,k)}}\right)^{(r,k)}\right)\\&=\sum_{k\geq 1}(r,k)V_{k/(r,k)}\underline{S(a_k^{r/(r,k)})}\\&=\sum_{k\geq 1}(r,k)V_{k/(r,k)}F_{r/(r,k)}\underline{S(a_k)}\quad\text{from 5.7 (ii) of [3]}\\&=\sum_{k\geq 1}F_rV_k\underline{S(a_k)}\text{ from 5.5 of [3]}\\&=F_r\left(\sum_{k\geq 1}V_k\underline{S(a_k)}\right).\end{aligned}$$

This proves (35).

Since $\chi:\mathrm{W}(A)\to\mathrm{Ap}(A)$ preserves V_r and F_r it follows that $\mathrm{D}(A)=\mathrm{Image}\ \chi$ is invariant under V_r and F_r for all $r\geq 1$.

To complete the proof of Theorem 2.2 we have only to show that $\phi\chi:\mathrm{W}(A)\to\mathrm{Gh}(A)$ is injective if and only if $A\in\mathscr{A}$.

First, assume that $A\in\mathscr{A}$. We know that for any $\mathbf{a}=(a_1,a_2,a_3,\ldots)$ in $\mathrm{W}(A)$, $\phi\chi(\mathbf{a})=\mathbf{g}=(g_1,g_2,g_3,\ldots)\in\mathrm{Gh}(A)$ where $g_n=\sum_{d|n}da_d^{n/d}$. When $g_n=0$ for all $n\geq 1$, we need to show that $a_n=0$ for all $n\geq 1$. Now $g_1=a_1$. Hence $a_1=0$. Let $n\geq 2$. Assuming by induction we have shown that $a_1=\cdots=a_{n-1}=0$. Then $g_n=na_n$. Thus $g_n=0\Rightarrow na_n=0$. Since $A\in\mathscr{A}$ we get $a_n=0$.

Conversely, assume that $A\notin\mathscr{A}$. We will show that $\phi\chi:\mathrm{W}(A)\to\mathrm{Gh}(A)$ is not injective in this case. Since $A\notin\mathscr{A}$, $\exists$ an integer $n\geq 2$ and an element $b\neq 0$ in A with $nb=0$.

Let $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in \mathrm{W}(A)$ be given by $a_i = 0$ for $i \neq n$ and $a_n = b$. Then it is easily seen that $\phi\chi(\mathbf{a}) = 0 \in \mathrm{Gh}(A)$, proving that $\phi\chi$ is not injective. This completes the proof of Theorem 2.2. □

COROLLARY 2.3

Let $A \in \mathscr{A}$ and $A_0 = A \underset{z}{\otimes} Q$ the rationalization of A. Then $\Delta(A) = \{\sum_{k \geq 1} V_k \underline{M(a_k)} \mid a_k \in A$ for all $k \geq 1\}$ is a subring of $\mathrm{Nr}(A_0)$ invariant under V_r and F_r for all $r \geq 1$. The map $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in \mathrm{W}(A) \mapsto \sum_{k \geq 1} V_k \underline{M(a_k)}$ defines an isomorphism of the ring $\mathrm{W}(A)$ with $\Delta(A)$ preserving V_r and F_r for all $r \geq 1$.

Proof. Immediate consequence of Theorem 2.2 and the fact that $\psi: \mathrm{Nr}(A_0) \to \mathrm{Ap}(A_0)$ is a ring isomorphism satisfying

$$\psi\left(\sum_{k \geq 1} V_k \underline{M(a_k)}\right) = \sum_{k \geq 1} V_k \underline{S(a_k)}$$

for any sequence $\{a_k\}_{k \geq 1}$ of elements in A_0.

References

[1] Michiel Hazewinkel, *Formal groups and applications* (New York: Academic Press) (1978)

[2] Metropolis N and Gian Carlo Rota, Witt vectors and the algebra of neckaces, *Adv. Math.* **50**(1983) 95–125

[2] Varadarjan K and Wehrhahn K, A periodic rings, necklace rings and Witt vectors (to appear)

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Classification of isolated complete intersection singularities

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Abstract. In this article we prove that an isolated complete intersection singularity $(V, 0)$ is characterized by a module of finite length $A(V)$ (cf. §1 for definition) associated to it. The proof uses the theory of finitely determined map germs and generalises the corresponding result by Yau and Mather [4], for hypersurfaces.

Keywords. Intersection singularity; finite length; map germs.

Statement of the results

We use the following notation.
$(\mathbb{C}^m, 0)$:Germ at the origin of $\mathbb{C}^m$.

$\mathcal{O}_m$:Ring of germs of holomorphic functions on $(\mathbb{C}^m, 0)$.

$\mathfrak{M}_n$:Maximal ideal of $\mathcal{O}_m$.

Ω_m:$\mathcal{O}_m$-module of Kähler differentials of order 1

Θ_m:$\mathbb{C}$-derivations of $\mathcal{O}_m$ (tangent sheaf of $(\mathbb{C}^m, 0)$).

In [1] Le and Ramanujam had proved that the moduli algebra determines the topological type of an isolated hyper surface singularity. In [4] Mather and Yau proved that an isolated hyper surface singularity is characterized by its dimension and the moduli algebra.

Here we prove the analogous result for an isolated complete intersection singularity, i.e. an isolated complete intersection singularity, $(V, 0)$, is characterized by its dimension and the $\mathcal{O}_V$-module $\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$.

After this work was done, the paper [5] was brought to my notice, in which a different method is used to give a classification of singularities of isolated type, this includes the case considered in this paper. However the present method, which follows that of Yau and Mather, involves explicit computations (see §4) and may be of independent interest in dealing with certain related problems.

Remark. Our method can be extended to yield the stronger result that if $(V, 0)$ is any analytic germ with no smooth curve contained in the singular locus, then the

isomorphism class of the module $A(V)$ defined by the exact sequence (#) [not in general equal to $\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$] characterises the analytic isomorphism type of V. This result is also contained in [5].

Let $f:(\mathbb{C}^m,0)\to(\mathbb{C}^k,0)$ be a holomorphic map-germ. Let $\Theta(f)$ be the pull-back of the sheaf of germs of holomorphic vector fields at the origin of $(\mathbb{C}^k,0)$ i.e. $\Theta(f)=\mathcal{O}_m\otimes_{f^*\mathcal{O}_k}\Theta_k$. $\partial f:\Theta_m\to\Theta(f)$ be the derivative map. $V=(f_1,\ldots,f_k)$ be the analytic space defined by $f=0$. Set

$$A(V)=\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V,\mathcal{O}_V)$$

$$J(V)=\mathrm{Ann}_{\mathcal{O}_V}\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V,\mathcal{O}_V).$$

A presentation for $(V,0)\subset(\mathbb{C}^m,0)$ is a choice of generators for $I(V)$, the ideal defining $(V,0)$ in $(\mathbb{C}^m,0)$. This is equivalent to giving a map germ $f:(\mathbb{C}^m,0)\to(\mathbb{C}^k,0)$ for some k such that $\{f=0\}=(V,0)$ in $(\mathbb{C}^m,0)$. Let $(W,0)=V(g_1\ldots g_k)$ be the analytic space defined by $g=0$. Assume that $(V,0)$ and $(W,0)$ are the isolated complete intersection singularities of dimension n, and $\mathcal{O}_V$ and $\mathcal{O}_W$ the corresponding analytic local rings. Then we prove the following:

Main Theorem. *Suppose there exists a $\mathbb{C}$-algebra isomorphism h: $\mathcal{O}_V/J(V)\to\mathcal{O}_W/J(W)$ such that $A(V)\approx h^*A(W)$ i.e. there exists an abelian group isomorphism ψ: $A(V)\to A(W)$ such that $\psi(a,m)=h(a)\cdot\psi(m)$ for all $m\in A(V)$ and $\forall a\in\mathcal{O}_V/J(V)$. Then $\mathcal{O}_V\xrightarrow{\approx}\mathcal{O}_W$ as $\mathbb{C}$-algebras.*

Notice that for any presentation of $(V,0)$ as an isolated complete intersection singularity germ $(V,0)=V(f_1\ldots f_k)$ in $(\mathbb{C}^m,0)$, we have an exact sequence,

$$0\to f^*(\mathfrak{M}_k)\Theta(f)+\partial f(\Theta_m)\to\Theta(f)\to A(V)\to 0. \tag{\#}$$

Observe that $\Theta(f)$ is a free $\mathcal{O}_m$-module with basis $1\otimes(\partial/\partial t_i)$. Hence $A(V)$ can be considered as an $\mathcal{O}_{n+k}$-module. The following lemmas are standard.

Lemma 1. If $(V,0)$ is a complete intersection, then the minimal number of generators of $A(V)$ as an $\mathcal{O}_V$-module, $\mu(A(V))$, equals the embedding codimension of V. Thus the embedding dimension equals $\mu(A(V))+\dim V$.

Lemma 2. Let R and S be local rings which are quotients of $\mathcal{O}_m$. Then any isomorphism of $\mathbb{C}$-algebras $Q:R\to S$ can be lifted to an automorphism of $\mathcal{O}_m$ as a $\mathbb{C}$-algebra.

COROLLARY 1

Let $(V,0)$ and $(W,0)$ be as in the main theorem. Then we can embed $(V,0)$ and $(W,0)$ in $(\mathbb{C}^{n+k},0)$ with $n=\dim(V,0)$ and $k=\mu(A(V))$, such that $A(V)$ is isomorphic to $A(W)$ as $\mathcal{O}_{n+k}$-modules.

Proof. By the hypothesis of the theorem $m\mu(A(V))=\mu(A(W))$ both have the same embedding dimension by Lemma 1. By Lemma 2, the isomorphism, h: $\mathcal{O}_V/J(V)\to\mathcal{O}_W/J(W)$, can be lifted to an automorphism $\tilde{h}:\mathcal{O}_{n+k}\hookleftarrow$. By replacing $(W,0)$ by its image i.e. $I(W)$ by h (I(W)), we can assume that $A(V)$ is actually isomorphic to $A(W)$ as an $\mathcal{O}_{n+k}$ module.

2. The Group $\mathscr{K}$ and a theorem of Mather

In this section we recall some definitions and a theorem from [2].

Elements of $\mathscr{K}$ are pairs (h, H) of holomorphic automorphisms $h:(\mathbb{C}^{n+k},0)\to(\mathbb{C}^{n+k},0)$ and $H:(\mathbb{C}^{n+2k},0)-(\mathbb{C}^{n+2k},0)$, such that the following diagram commutes:

$$\begin{array}{ccccc}
(\mathbb{C}^{n+k},0) & \xrightarrow{i} & (\mathbb{C}^{n+2k},0) & \xrightarrow{\pi} & (\mathbb{C}^{n+k},0)\\
\downarrow h & & \downarrow H & & \downarrow h\\
(\mathbb{C}^{n+k},0) & \xrightarrow{i} & (\mathbb{C}^{n+2k},0) & \xrightarrow{\pi} & (\mathbb{C}^{n+k},0).
\end{array}$$

Here i and π are $i(z_1,\dots,z_{n+k})=(z_1,\dots,z_{n+k},0,\dots,0)$ and $\pi(z_1,\dots,z_{n+2k})=(z_1,\dots,z_{n+k})$. We define

$$\mathscr{F}=\{f:(\mathbb{C}^{n+k},0)\to(\mathbb{C}^k,0)/f \text{ is holomorphic}\}.$$

We define an action of $\mathscr{K}$ on $\mathscr{F}$ as follows. Let $(h,H)\in\mathscr{K}$ and $f\in\mathscr{F}$, then there exist a unique $g\in\mathscr{F}$ such that graph $g=H(\text{graph } f)$. We set $(h,H)\cdot f=g$.

Lemma 3. If f and g define n-dimensional spaces $(V,0)$ and $(W,0)$ then they are in the same $\mathscr{K}$-orbit if and only if $(V,0)$ and $(W,0)$ are biholomorphically equivalent.

Proof. Let $(h,H)\in\mathscr{K}$ be such that $(h,H)\cdot f=g$. Then $h^{-1}(W)=h^{-1}\circ i^{-1}(\text{graph } g)=i^{\ 1}(H^{-1}(\text{graph } g))=i^{-1}$ (graph $f)=V$. Hence h: $(\mathbb{C}^{n+k},0)\to(\mathbb{C}^k,0)$ provides a biholomorphic equivalences between $(V,0)$ and $(W,0)$.

Conversely suppose $h:(\mathbb{C}^{n+k},0)\to(\mathbb{C}^{n+k},0)$ is a biholomorphic map such that $h(V)=W$. Then we define H: $(\mathbb{C}^{n+2k},0)\to(\mathbb{C}^{n+2k},0)$ as $H(z,w)=(h(z),\ U^{-1}(z),\ w)$, where $U\in GL(k,\mathcal{O}_{n+k})$ is such that $f=U(g\circ h)$, for $z\in\mathbb{C}^{n+k}$ and $w\in\mathbb{C}^k$. Then clearly $(h,H)\in\mathscr{K}$. Also,

$$\begin{aligned}H(\text{graph } f)&=H(z,f(z))=(h(z),U^{-1}(z)f(z))=(h(z),g\circ h(z))\\&=(h(z),g(h(z))=\text{graph } g.\end{aligned}$$

Hence $(h,H)\cdot f=g$.

Let $\mathscr{K}_r$ be the subgroup of $\mathscr{K}$ consisting of all $(h,H)\in\mathscr{K}$ such that its r-jet at 0, is the same as the r-jet of the identity. $\mathscr{K}^{(r)}=\mathscr{K}_0/\mathscr{K}_r$. Then $\mathscr{K}^{(r)}$ is a Lie group. Let $\mathscr{F}^r$ be the r-jet of elements of $\mathscr{F}$. Then $\mathscr{K}^{(r)}$ will act on $\mathscr{F}^r$ by $(h,H)^{(r)}\cdot f^{(r)}=((h,H)\cdot f)^{(r)}$.

An element $f\in\mathscr{F}$ is said to be r-determined relative to $\mathscr{K}$ if for any $g\in\mathscr{F}$ with $g^{(r)}\in\mathscr{K}^{(r)}\cdot f^{(r)}$, $g\in\mathscr{K}\cdot f$. An element of $\mathscr{F}$ is said to be finitely determined relative to $\mathscr{K}$ if it is r-determined for some $r\in\mathbb{N}$. In this situation we have

Theorem 1. ([2], Theorem 3.5): *$f\in\mathscr{F}$ is finitely determined if and only if $\Theta(f)/\partial f(\Theta_{n+k})+f^*(\mathfrak{M}_k)\Theta(f)$ is of finite $\mathbb{C}$-dimension.*

Now by (#), the function, f defining the isolated complete intersection singularity is finitely determined, because $\mathrm{Ext}^1_{\mathcal{O}_V}(\Omega_V,\mathcal{O}_V)$ has finite length, where V is defined by $\{f=0\}$.

3. Reduction to a special case

From now on we fix coordinates $t_1, \ldots, t_k$ on $(\mathbb{C}^k, 0)$. Let $V = V(f_1, \ldots, f_k)$ be as in the main theorem.

Lemma 4. *Let $f'_i = \sum a_{ij} f_j$ $1 \leqslant i, j \leqslant k$, be another set of generators for the ideal $I(V)$. Let $\rho_i = 1 \otimes dt_i \in \mathcal{O}_{n+k} \otimes_{f^*\mathcal{O}_k} \Omega_k$ and $\rho'_i = 1 \otimes dt_i \in \mathcal{O}_{n+k} \otimes_{f'^*\mathcal{O}_k} \Omega_k$ be the free basis for the corresponding $\mathcal{O}_{n+k}$-modules. Then we have the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{O}_V \otimes (\mathcal{O}_{n+k} \otimes_{f^*\mathcal{O}_k} \Omega_k) \arrow[d] \arrow[dr, "1 \otimes U"] & \\
I/I^2 \arrow[r] & \mathcal{O}_V \otimes (\mathcal{O}_{n+k} \otimes_{f'^*\mathcal{O}_k} \Omega_k)
\end{tikzcd}$$

Here the vertical isomorphism is given by $1 \otimes \rho_i \mapsto f_i$ and the horizontal isomorphism is given by $1 \otimes \rho'_i \mapsto f_i$ and U is defined by

$$U(\rho'_i) = \sum a_{ij} \rho_j.$$

Proof. The commutativity of the diagram follows from the definition of those maps and a simple diagram chasing.

Now notice that

$$\Theta(f) = \mathrm{Hom}_{\mathcal{O}_{n+k}} (\mathcal{O}_{n+k} \otimes_{f^*\mathcal{O}_k} \Omega_k, \mathcal{O}_{n+k}) \text{ and}$$

$$\Theta(g) = \mathrm{Hom}_{\mathcal{O}_{n+k}} (\mathcal{O}_{n+k} \otimes_{g^*\mathcal{O}_k} \Omega_k, \mathcal{O}_{n+k}).$$

Hence we have the following

COROLLARY 2

U induces an isomorphism $U^t: \Theta(f) \to \Theta(f')$, which induces the identity on $A(V) = \mathrm{Ext}'_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$.

Lemma 5. *Let $(V, 0)$ and $(W, 0)$ in $(\mathbb{C}^{n+k}, 0)$ be as in Corollary 1. Then any given presentation $g = (g_1, \ldots, g_k)$ of $(W, 0) \subset (\mathbb{C}^{n+k}, 0)$ we can choose a presentation $f = (f_1, \ldots, f_k)$ of $(V, 0) \subset (\mathbb{C}^{n+k}, 0)$ such that the isomorphism $A(V) \xrightarrow{\sim} A(W)$ of Corollary 1 can be lifted to an isomorphism $\Theta(f) \to \Theta(g)$ such that*

$$\frac{\partial}{\partial t_i} \circ f \mapsto \frac{\partial}{\partial t_i} \circ g.$$

Proof. By Nakayama's lemma we can get a lift $U: \Theta(f) \to \Theta(g)$, of $A(V) \to A(W)$. Let $U = (a_{ij})$ be an invertible $k \times k$ matrix with entries in $\mathcal{O}_{n+k}$ such that $U^t = \tilde{U}$. Here $\tilde{U}$ is thought of as a matrix with respect to the basis of $\Theta(f)$ and $\Theta(g)$ given by $(\partial/\partial t_i) \circ f$ and $(\partial/\partial t_i) \circ g$ respectively. Then by, Corollary 2, if we change the generators of $I(V)$ by $f'_i = \sum a_{ij} f_j$, then $U^{t^{-1}}: \Theta(f') \to \Theta(f)$ is an isomorphism in inducing identity on $A(V)$. Hence if we replace f by f', the isomorphism $\Theta(f') \to \Theta(g)$ has the required property.

4. Some local analytic computations

In this section we prove Proposition 6, which is the main technical step in the proof of the main theorem. From now on, we fix the presentations f, g for $(V,0)$; $(W,0)$, respectively, given by Lemma 5. Let $\epsilon_i = (\partial/\partial t_i)f$ and $\epsilon_i' = (\partial/\partial t_i)g$ be the natural basis vectors for $\Theta(f)$ and $\Theta(g)$ respectively. Then under the identification of $\Theta(f)$ and $\Theta(g)$ with $\mathcal{C}^k_{n+k}$ given by the basis ϵ_i and ϵ_i', the following equality

$$\partial f(\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f) = \partial g(\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g) \qquad (*)$$

obtained from (#) of §1 takes the following explicit form (once we fix coordinates $z_1,\ldots,z_{n+k}$ on $(\mathbb{C}^{n+k},0)$): The L.H.S. (left-hand side) of $(*)$ is generated by the vectors

$$\left(\frac{\partial f_1}{\partial z_i},\ldots,\frac{\partial f_k}{\partial z_i}\right) \quad i = 1,\ldots,n+k, \text{ and } (f_i,0\ldots0),$$

$(0f_i,0\ldots0),\ldots,(0,0,\ldots,f_i)i = 1,\ldots,k$. Similarly the R.H.S. (right-hand side) is generated by $((\partial g_1/\partial z_i),\ldots,(\partial g_k/\partial z_i))$, and

$$(g_j,0\ldots0),\ldots,(0,\ldots,0,g_j)\ 1 \le i \le n+k$$
$$1 \le j \le k.$$

In this situation we have the following:

PROPOSITION 6

$\partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f) = \partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$.

Proof. Since the proofs are identical we prove only one inclusion,

L.H.S. $\subset$ R.H.S.

Step 1. $\partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) \subset \partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$.

Proof. Notice that element of $\partial f(\mathfrak{M}_{n+k}\Theta_{n+k})$ are generated by

$$z_i\left(\frac{\partial f_1}{\partial z_j},\ldots,\frac{\partial f_k}{\partial z_j}\right), \quad 1 \le i,j \le n+k.$$

Since by $(*)$ $((\partial f_1/\partial z_j),\ldots,(\partial f_k/\partial z_j)) \in \partial g(\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$ and ∂g is $\mathcal{C}_{n+k}$ linear, we have

$$z_i\left(\frac{\partial f_1}{\partial z_j},\ldots,\frac{\partial f_k}{\partial z_j}\right) \in \partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$$

Step 2. $f^*(\mathfrak{M}_k)\Theta(f) \subset \partial g(\mathfrak{M}_{n+k}\Theta_{n+k}) + g^*(\mathfrak{M}_k)\Theta(g)$.

Proof. Here $f^*(\mathfrak{M}_k)\Theta(f)$ is generated by $(f_i,0\ldots0),\ldots,(0,0\ldots f_i)$ $1 \le i \le k$. So we have to prove that all these k^2 elements belong to the R.H.S. Again since the argument is

identical, we prove that $(f_1,0,\ldots,0)\in$R.H.S. By $(*)$ $(f_1,0\ldots 0)\in\partial g(\Theta_{n+k})+g^*(\mathfrak{M}_k)\Theta(g)$ i.e. $(f_1,0\ldots 0)=v+\partial g(\eta)$ for some $v\in g^*(\mathfrak{M}_k)\Theta(g)$ and $\eta\in\Theta_{n+k}$.

Key Lemma. $\eta\in\mathfrak{M}_{n+k}\Theta_{n+k}$.

Proof. Suppose $\eta(0)\neq 0$. Then we can choose a coordinate system $z_1,\ldots,z_{n+k}$ in $\mathbb{C}^{n+k}$ such that $\eta=(\partial/\partial z_1)$. Then the above equation will become

$$(f_1,0,\ldots,0)=v+\partial g\left(\frac{\partial}{\partial z_1}\right)$$

i.e. by Lemma 5, we have the following equations;

$$\begin{aligned}
&(1)\ f_1=a_1^{(1)}g_1+a_2^{(1)}g_2+\cdots+a_k^{(1)}g_k+\frac{\partial g_1}{\partial z_1}\\
&(2)\ \ 0=a_1^{(2)}g_1+a_2^{(2)}g_2+\cdots+a_k^{(2)}g_k+\frac{\partial g_2}{\partial z_1}\\
&(k)\ \ 0=a_1^{(k)}g_1+a_2^{(k)}g_2+\cdots+a_k^{(k)}g_k+\frac{\partial g_k}{\partial z_1}
\end{aligned}\qquad (**)$$

$$a_i^{(j)}\in\mathcal{O}_{n+k}.$$

Also by $(*)$ $(\partial f/\partial z_j)\in\partial g\ (\Theta_{n+k})+g^*(\mathfrak{M}_k)\Theta(g)$. Looking at the first component (coefficients of $e_1=e_1'$) of $\partial f(\partial/\partial z_j)$, we obtain

$$\frac{\partial f_1}{\partial z_j}=b_1g_1+\cdots+b_kg_k+\sum_{i=1}^{n+k}\frac{\partial g_1}{\partial z_i}\xi_i \qquad (***)$$

DEFINITION

For any $h\in\mathcal{O}_{n+k}$, let $(h)_\nu$ denote the νth-order homogeneous component of h with respect to the coordinate system chosen above. Let $o(h)$ be the order of h. Let $\deg_{z_1}(h)_\nu$ be the degree of $(h)_\nu$ as a polynomial in z_1. Let $m=\mathrm{Min}_{1\leq i,j\leq k}\{o(f_i),o(g_j)\}$. Then by the assumption that k is the embedding codimension for both $(V,0)$ and $(W,0)$, we have $m\geq 2$. Notice that

$$(f_1)_{m-1}=(g_1)_{m-1}=\cdots=(g_k)_{m-1}=0.$$

Claim (i). $(\partial g_r/\partial z_1)_{m-1}=0$ i.e. $\deg_{z_1}(g_r)_m=0$ for $r=1,\ldots,k$.

Proof. By (r) of $(**)$, $(\partial g_r/\partial z_1)_{m-1}=0$ hence $\deg_{z_1}(g_r)_m=0$.

Claim (ii). $\deg_{z_1}(f_1)_m=0$.

Proof. For it is enough to check that $\deg_{z_1}(\partial f_1/\partial z_j)_{m-1}=0$, for every j, because $m\geq 2$. But then by looking at $(***)$ we find that

$$\left(\frac{\partial f_1}{\partial z_j}\right)_{m-1} = (b_1 g_1)_{m-1} + \cdots + (b_k g_k)_{m-1} + \left(\sum \frac{\partial g_1}{\partial z_i}\xi_i\right)_{m-1}$$
$$= \sum \left(\frac{\partial g_1}{\partial z_i}\right)_{m-1} \xi_i(0).$$

Now since $\deg_{z_1}(g_1)_m = 0$, we have $\deg_{z_1}(\partial g_i/\partial z_i)_{m-1} = 0$ and hence $\deg_{z_1}(\partial f_1/\partial z_j)_{m-1} = 0$. Hence $\deg_{z_1}(f_1)_m = 0$. Now we prove the following assertions by induction on v. The argument above yields the case $v = 0$, to begin the induction.

Assertion $(i)_v$. $\deg_{z_1}(g_1)_{m+v}, \ldots, \deg_{z_1}(g_k)_{m+v} \leqslant v$. i.e. none of these homogeneous polynomials contain monomials of degree bigger than v, in z_1.

Assertion $(ii)_v$. $\deg_{z_1}(f_1)_{m+v} \leqslant v$.

Proof of $(i)_v$. We assume $(\mathrm{i})_\mu$ and $(\mathrm{ii})_\mu$ for all $\mu < v$. i.e. $\deg_{z_1}(g_i)_{m+\mu}$, $\deg_{z_1}(f_1)_{m+\mu} \leqslant \mu$ for all $\mu < v$. Let $D^\mu = (\partial^\mu/\partial z_1^\mu)$. Then $o(D^\mu g_i) \geqslant m$ and $o(D^\mu f_1) \geqslant m$ for every $\mu \leqslant v$. We prove that $\deg_{z_1}(D^v g_i)_m = 0$. Now apply D^v to both sides of (1) of (**),

$$D^v f_1 = D^v(a_1^{(1)} g_1) + \cdots + D^v(a_k^{(1)} g_k) + D^v \frac{\partial g_1}{\partial z_1}.$$

Since

$$D^v(a_i^{(1)} g_i) = \sum \binom{v}{r} D^r a_i^{(1)} D^{v-r} g_i$$

has order $\geqslant m$, we have

$$0 = (D^v f_1)_{m-1} = \left(D^v \frac{\partial g_1}{\partial z_1}\right)_{m-1} = \frac{\partial}{\partial z_1}(D^v g_1)_m.$$

Hence

$$\deg_{z_1}(D^v g_1)_m = 0 \text{ i.e., } \deg_{z_1}(g_1)_{m+v} \leqslant v.$$

Similarly by applying D^v to both sides of the other equations of (**) and comparing the $(m-1)$th order terms on both sides we obtain

$$0 = D^v\left(\frac{\partial g_i}{\partial z_1}\right)_{m-1} = \frac{\partial}{\partial z_1}(D^v g_i)_m.$$

Hence $\deg_{z_1}(D^v g_i) = 0$ i.e. $\deg_{z_1}(g_i)_{m+v} \leqslant v$. We deduce that $\deg_{z_1} D^v(\partial g_1/\partial z_1)_{m-1} = 0$.

Proof of $(ii)_v$. Apply D^v to the equation (***),

$$D^v \frac{\partial f_1}{\partial z_j} = \frac{\partial}{\partial z_j}(D^v f_1) = D^v(b_1 g_1) + \cdots + D^v(b_k g_k) + D^v \sum \frac{\partial g_1}{\partial z_i}\xi_i.$$

Notice that $o\,(D^v(b_i g_i)) \geqslant m$. So again by comparing the terms of order $m-1$, we get

$$\left(\frac{\partial}{\partial z_j} D^v f_1\right)_{m-1} = \left(\sum D^v\left(\frac{\partial g_1}{\partial z_i}\xi_i\right)\right)_{m-1}$$

$$=\left(\sum\sum\binom{\mu}{r}D^r\frac{\partial g_1}{\partial z_i}D^{\nu-r}\xi_i\right)_{m-1}$$

$$=\sum\sum\binom{\nu}{r}\left(D^r\frac{\partial g_1}{\partial z_i}\right)_{m-1}D^{\nu-r}\xi_i(0).$$

Since $\deg_{z_1} D^r(\partial g_i/\partial z_i)=0$, for every $r\leqslant\nu$, we conclude that $\deg_{z_1}((\partial/\partial z_j)D^\nu f_1)_{m-1}=0$. Hence $\deg_{z_1}(f_1)_{m+\nu}\leqslant\nu$. This proves (ii)$_\nu$. Since we have already checked the induction hypothesis for $\nu=0$, we get, $\deg_{z_1}(g_i)_{m+\nu}\leqslant\nu$ for all ν.

Now since $m\geqslant 2$, every monomial of g_i is of order $\geqslant 2$ in the variables $z_2,\dots,z_{n+k}$. Hence $g_i(z_1,0,\dots,0)=0$ and $(\partial g_i/\partial z_j)\ (z_1,0,\dots,0)=0$ for every i and j. Hence the Jacobian matrix $(\partial g_i/\partial z_j)$ is identically zero along the z_1-axis, which implies that z_1-axis is contained in the singular locus of $\{g=0\}$. This contradicts the assumption that g defines an isolated singularity. This contracdiction was due to the assumption that $\eta(0)\neq 0$. Hence $\eta(0)=0$ i.e. $\eta\in\mathfrak{M}_{n+k}\Theta_{n+k}$.

5. Proof of the Main Theorem

We fix presentations $\{f=0\}$ and $\{g=0\}$ for $(V,0)$ and $(W,0)$ as in Lemma 5. Then by Theorem 1, f and g are finitely determined. By Lemma 3 f and g are biholomorphically equivalent if and only if they are in the same $\mathscr{K}$-orbit. But to prove f and g are in the same $\mathscr{K}$-orbit it is enough to prove $f^{(l)}$ and $g^{(l)}$ are in the same $\mathscr{K}^{(l)}$-orbit, for every l. Now fix an $l\in\mathbb{N}$. Note that $\mathscr{F}^{(l)}$ can be given a global coordinate system so that it has the structure of a complex affine space. In his paper [2]. Mather defines a projection, π^l: $\mathfrak{M}_{n+k}\Theta(h)\to T_{h^{(l)}}\mathscr{F}^{(l)}$, for any $h\in\mathscr{F}$. Here $T_{h^{(l)}}\mathscr{F}^{(l)}$ is the tangent space to $\mathscr{F}^{(l)}$ at the l-jet $h^{(l)}$ of h. In our context, if we identify $T_{h^{(l)}}\mathscr{F}^{(l)}$ with $\mathscr{F}^{(l)}$ (using the affine structure), and $\Theta(h)$ with $\mathcal{O}^k_{n+k}$ using the basis $(\partial/\partial t_i)\circ h$, then for any $\eta=(\eta_1,\dots,\eta_k)\in\mathfrak{M}_{n+k}\Theta(h)$ $(\approx\mathfrak{M}_{n+k}\mathcal{O}^k_{n+k})$, $\pi^l(\eta)=\eta^{(l)}$. We may think of η as an element of $\mathscr{F}\cdot\mathscr{F}^{(l)}$ has a natural structure as an $\mathcal{O}_{n+k}$ module; note that π^l is then $\mathcal{O}_{n+k}$-linear. Mather also proves the following:

Theorem 2. ([2], Proposition 7.4).

$$T_{h^{(l)}}(\mathscr{K}^{(l)}\cdot h^{(l)})=\pi^l(\partial h(\mathscr{M}_{n+k}\Theta_{n+k})+h^*(\mathfrak{M}_k)\Theta(h)).$$

Here $T_h(l)\ \mathscr{K}^{(l)}\cdot h^{(l)}$ denotes the tangent space to the Space to the $\mathscr{K}^{(l)}$-orbit of $h^{(l)}$ at the point $h^{(l)}\in\mathscr{F}^{(l)}$.

Note that by Theorem 2, $T_h(l)\mathscr{K}^{(l)}h^{(l)}$ is an $\mathcal{O}_{n+k}$-submodule of $T_h(l)\ \mathscr{F}^{(l)}$ and it is generated as a module by the elements of the form $(z_j(\partial h_1/\partial z_i),\dots,z_j\ (\partial h_k/\partial z_i))^{(l)}$ and $(h_i,0\dots0)^{(l)},\dots,(0,0,\dots,h_i)^{(l)}$. We denote these generators by

$$\rho_1(h),\dots,\rho_N(h),\ \text{where}\ N=k^2+(n+k)^2.$$

If $f^{(l)}\neq g^{(l)}$, then consider the complex line L joining $f^{(l)}$ and $g^{(l)}$. Define

$$\begin{aligned}L_0&=\{h^{(l)}\in L/\pi^l(\partial h(\mathfrak{M}_{n+k}\Theta_{n+k})+h^*(\mathfrak{M}_k)\Theta(h))\\&=\pi^l(\partial f(\mathfrak{M}_{n+k}\Theta_{n+k})+f^*(\mathfrak{M}_k)\Theta(f))\}\\&=\{h^{(l)}\in L/T_{h^{(l)}}(\mathscr{K}^{(l)}\cdot h^{(l)})=T_f(l)(\mathscr{K}^{(l)}\cdot h^{(l)})\}.\end{aligned}$$

Then L_0 has the following properties:

(1) $g^{(l)} \in L_0$ by Proposition 6.
(2) $h^{(l)} \in L$, $h^{(l)} = (1-t)f^{(l)} + tg^{(l)}$, then $\rho_i(h) = (1-t)\rho_i(f) + t\rho_i(g)$. Hence $T_{h^{(l)}}(\mathscr{K}^{(l)} \cdot h^{(l)}) \subset T_{f^{(l)}}(\mathscr{K}^{(l)} \cdot f^{(l)})$.
(3) $\{\rho_1(f), \ldots, \rho_N(f)\}$ and $\{\rho_1(g), \ldots, \rho_N(g)\}$ generate the same $\mathcal{O}_{n+k}$-submodule in $\mathscr{F}^{(l)}$, hence for all but a finite set of $t \in \mathbb{C}$, $\{(1-t)\rho_i(f) + t\rho_i(g)\}$ will generate the same submodule.
(4) By (2) and (3), L_0 is connected, since L_0 is $\mathbb{C}$ with at most finitely many points deleted.

Now we prove that L_0 is contained in a single orbit of $\mathscr{K}^{(l)}$. For this we need the following theorem,

Theorem 3 ([3], Lemma 3.1). *Let α: $G \times U \to U$ be a C^∞ action of a Lie group G on a C^∞ manifold U, $V \subset U$ a connected submanifold of U. Then V is contained in a single orbit of α if and only if*

(i) $T_v V \subset T_v G \cdot v$
(ii) $\dim T_v G \cdot v$ *is independent of* $v \in V$.

In our context we take $G = \mathscr{K}^{(l)}$, $U = \mathscr{F}^{(l)}$, $V = L_0$ and check the conditions (i) and (ii) of Theorem 3. Notice that $T_{h^{(l)}} L_0$ is generated by $(f-g)^{(l)}$, for any $h^{(l)} \in L_0$. By Proposition 6, $f - g \in \partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f)$. Hence

$$
\begin{aligned}
(f-g)^{(l)} &\in \pi^l(\partial f(\mathfrak{M}_{n+k}\Theta_{n+k}) + f^*(\mathfrak{M}_k)\Theta(f)) \\
&= \pi^l(\partial h(\mathfrak{M}_{n+k}\Theta_{n+k}) + h^*(\mathfrak{M}_k)\Theta(h)) \\
&= T_{h^{(l)}}\mathscr{K}^{(l)} \cdot h^{(l)}.
\end{aligned}
$$

Hence

$$T_{h^{(l)}} L_0 \subset T_{h^{(l)}}\mathscr{K}^{(l)} \cdot h^{(l)} \quad \forall h^{(l)} \in L_0.$$

Since $T_{h^{(l)}}\mathscr{K}^{(l)} \cdot h^{(l)} = T_{f^{(l)}}\mathscr{K}^{(l)} \cdot f^{(l)}$, $\dim T_{h^{(l)}}\mathscr{K}^{(l)}$ is independent of $h^{(l)} \in L_0$. Hence (i) and (ii) are satisfied, and so L_0 is contained in a single orbit of $\mathscr{K}^{(l)}$. In particular $f^{(l)} \in \mathscr{K}^{(l)} \cdot g^{(l)}$. Hence $f \in \mathscr{K} \cdot g$. This completes the proof of the main theorem.

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References

[1] Le D T and Ramanujam C P, The invariance of Milnor's number implies the invariance of the topological type, *Am. J. Math.* **98** (1976) 67–78
[2] Mather J N, Stability of C^∞ mappings III, *Publ. Math. IHES* **35** (1968) 127–156
[3] Mather J N, Stability of C^∞ mappings IV, *Publ. Math. IHES* **37** (1970) 223–248
[4] Mather J N and Yau S S T, Classification of hyper surface singularities by their moduli algebras, *Invent. Math.* **69** (1982) 243–251
[5] Gaffney T and Hauser H, Characterizing singularities of varieties and of mappings. *Invent. Math.* **81** (1985) 427–447

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 1, April 1989, pp. 27–48.

Quasiminimal invariants for foliations of orientable closed surfaces*

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Abstract. The Katok bound for the dimension of the cone of invariant measures for "quasiminimal" orientable foliations of closed oriented surfaces is extended to the non-quasiminimal case, in particular allowing for more general singularities. Equivalence of the Katok bound with a bound for the dimension of the cone of invariant measures for a minimal interval exchange is established.

Keywords. Quasiminimal invariants; orientable foliations; closed oriented surfaces; non-quasiminimal invariants; Katok bound; P-triples; weakly minimal sets.

1. Introduction

Let M^2 be an oriented closed surface, and let $\mathfrak{F}$ be an oriented C^1 foliation of M^2 with finite singular set $\Sigma(\mathfrak{F})$. There exists a set $\mathscr{L}(\mathfrak{F})$ of C^1 flows on M^2 each of which has the elements of $\Sigma(\mathfrak{F})$ for stationary points and the leaves of $\mathfrak{F}$ for nontrivial orbits. Each $L \in \mathscr{L}(\mathfrak{F})$ has a cone $\mathscr{I}$ of finite *nonatomic* invariant Borel measures, and as was noted by Katok [4], when $\Sigma(\mathfrak{F}) \neq \phi$, it is generally the case that $\dim \mathscr{I}(L)$ is not constant on $\mathscr{L}(\mathfrak{F})$ (it is nonconstant if and only if $0 < \dim \mathscr{I} < \infty$ for some $L \in \mathscr{L}(\mathfrak{F})$). On the other hand one may also speak of the cone of nonatomic finite $\mathfrak{F}$-invariant measures (cf. [1]). For some $L \in \mathscr{L}(\mathfrak{F})$ there is a natural inclusion $\mathscr{I}(L) \to \mathscr{I}(\mathfrak{F})$ such that $\mathscr{I}(L)$ and $\mathscr{I}(\mathfrak{F})$ are isomorphic. We shall prove:

Theorem 1. *Assume each $\sigma \in \Sigma(\mathfrak{F})$ has only parabolic and/or hyperbolic sectors, and assume $\mathfrak{F}$ has but a finite number of closed leaves. Then*

$$\dim \mathscr{I}(\mathfrak{F}) \leqslant \operatorname{genus}(M^2).$$

Remark. In the case that $\mathfrak{F}$ is topologically transitive (whence Σ has only saddles) Theorem 1 is due to Katok.

We shall say a set $K \subseteq M^2$ is *quasiminimal* (for $\mathfrak{F}$) if

(a) K is closed and a union of leaves and singular points
(b) K contains a nonclosed leaf L such that

$$\bar{L}/L \nsubseteq \Sigma(\mathfrak{F})$$

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(c) For any leaf $L \in K$ satisfying (b),

$$\bar{L} = K.$$

Note that a quasiminimal set cannot be empty or a closed leaf. If K is a quasiminimal set, then K will support an element of $\mathscr{I}(\mathfrak{F})$, and conversely, as we shall see, any ergodic element of $\mathscr{I}(\mathfrak{F})$ has a quasiminimal set for its closed support.

We shall attach to every quasiminimal set K a pair $(M^2(K), \mathfrak{F}(K))$, where $M^2(K)$ is an oriented closed surface and $\mathfrak{F}(K)$ is a C^0 foliation of $M^2(K)$ for which $M^2(K)$ itself is a quasiminimal set. We set $g(K) = \text{genus}\, M^2(K)$. One connection between $\mathfrak{F}$, K and $\mathfrak{F}(K)$ is that $\mathscr{I}(\mathfrak{F}(K))$ is naturally isomorphic with the subcone of $\mathscr{I}(\mathfrak{F})$ consisting of elements with support K.

Theorem 2. *With notations as above*

$$\dim \mathscr{I}(\mathfrak{F}(K)) \leqslant g(K).$$

Given our construction, Theorem 2 follows directly from the aforementioned theorem of Katok [4]. Alternatively, it follows from a theorem in [8] on the dimension of the cone of invariant measures for a minimal "interval exchange." Indeed, our analysis will show that the latter result and that of Katok are equivalent.

Theorem 3. *Let $K_1, \ldots, K_p$ be pairwise distinct quasiminimal sets for $\mathfrak{F}$. Then*

$$\sum_{j=1}^{p} g(K_j) \leqslant \text{genus}(M^2).$$

One has $g(K_j) \geqslant 1$ for all j, and therefore Theorem 3 implies $p \leqslant \text{genus}(M^2)$. This fundamental result is due to A G Maier [6]. Maier's paper appears not to have been translated, and this paper may be regarded in part as an exposition of Maier's theorem.

If $\{K_1, \ldots, K_p\}$ is the set of all quasiminimal sets, the corresponding $\{(M^2(K_j), \mathfrak{F}(K_j)) \mid 1 \leqslant j \leqslant p\}$ are viewed as generalizing the rotation class of Poincaré for flows without singularities on the torus. Indeed we were motivated in this study by a class of real analytic flows on the torus constructed by Cherry [2]. For each flow in the class Σ has two elements, a node and a saddle of index -1. In addition each flow has a unique quasiminimal set (in the above sense) and Cherry's analysis amounts to the proposition $\mathfrak{F}(K)$ is an "irrational" foliation of the torus. In the general situation, if $(\mathbb{T}^2, \mathfrak{F})$ admits a quasiminimal set K (in the above sense), then K is unique, $g(K) = 1$, and $\mathfrak{F}(K)$ is an "irrational" foliation of $\mathbb{T}^2$.

2. *P*-triples

Below ∞ denotes an ideal point.

2.1 DEFINITION

A triple (X, D, T) shall be called a P-triple if it satisfies

(i) $X = X_1 \cup \cdots \cup X_d$, $d > 0$, where for each j X_j is homeomorphic to the unit circle in

ne. Each X_j is assumed to be oriented so that locally notions of "left" and "right" neaning

$\subsetneq X$ is a finite set

$:X \to X \cup \{\infty\}$ is right continuous, continuous on D^c

is one-to-one on $(T^{-1}\infty)^c$, orientation preserving on each component of $\emptyset)^c$.

rk 1. By (ii) and (iv) T has a left limiting value at any point of D. If S is T, ned at points of D to be the left limiting value, then $S:X \to X \cup \{\infty\}$ satisfies finition with "left" in place of "right".

rk 2. By (iv) T^{-1} is defined on $TX \cap X$. If we set $T^{-1}x = \infty$, $x \notin TX$, $D \cup SD$, then (X, D', T^{-1}) is a P-triple.

rk 3. If $x \in D$ is a point of discontinuity for T (and hence for S), then Tx and/or points of discontinuity for T^{-1} (and hence for S^{-1}) so long as they/it lies in X.

EFINITION

riple (X, D, T) is *aperiodic* if neither S nor T has a periodic point.

the most part we shall deal with aperiodic P-triples although the presence of e number of periodic points would not significantly alter the discussion.

EFINITION

$, D, T)$ be a P-triple, and let $\mathscr{S}$ be a nonempty collection of left closed-right open als in X. $\mathscr{S}$ is a *P-system* if

$\cap \bar{J} = \phi, I, J \in \mathscr{S}, I \neq J$.

$I \in \mathscr{S}$, then $T^{-1}I \subseteq J_1 \cup \cdots \cup J_p$ for some p and $J_1, \ldots, J_p \in \mathscr{S}$ (possibly $T^{-1}I = \phi$).

$I \in \mathscr{S}$, then $TI \cap X \subseteq L_1 \cup \cdots \cup L_q$ for some q and $L_1, \ldots, L_q \in \mathscr{S}$.

k. If in (b) (resp. (c)) T^{-1} (resp. T) is continuous on I, then by (a) we may take (resp. $q = 1$).

$\mathscr{S}$ be a P-system, and define $[x], x \in X$, by

$$[x] = \begin{cases} \bar{I} & x \in I \in \mathscr{S} \\ \{x\} & \text{otherwise} \end{cases} \tag{2.4}$$

well-defined because of (a). Since (2.4) partitions X into closed intervals, utomatic that each component of the quotient space is either a point or morphic to a circle. We assume the numbering is such that $X_1, \ldots, X_\delta$ project les $\tilde{X}_1, \ldots, \tilde{X}_\delta$, while $X_{\delta+1}, \ldots, X_d$ project to points. It is possible that $\delta = 0$ or below we shall assume $1 \leqslant \delta \leqslant d$. Let $\tilde{X} = \tilde{X}_1 \cup \cdots \cup \tilde{X}_\delta$, and let $\tilde{X}_j$ inherit its ation from X_j, $1 \leqslant j \leqslant d$.

$\in X_j$, $j \leqslant \delta$, then $[x] \neq X_j$ has a well-defined right hand endpoint w_x. The map is clearly right continuous.

2.5 Lemma. If $x \in X_j$, $j \leqslant \delta$, then

$$Tw_x = w_{Tw_x} \tag{2.6}$$

(Set $w_\infty = \infty$).

Proof. If $Tw_x = \infty$, there is nothing to prove. If $Tw_x \in X$, and if (2.6) fails, then there exists $I \in \mathscr{S}$ such that $Tw_x \in I$. But then $w_x \in T^{-1}I$, and by (b) $w_x \in J$ for some $J \in \mathscr{S}$. This implies $w_x \neq w_{w_x}$ which is absurd. The lemma is proved.

Next we define $\tilde{T}: \tilde{X} \to \tilde{X} \cup \{\infty\}$. Set $[\infty] = \{\infty\}$. If $[x] \in \tilde{X}$, define $[z] = \infty$, $z \in X_{j+1} \cup \cdots \cup X_d$ and

$$\tilde{T}[x] = [Tw_x].$$

The right continuity of $x \to w_x$ (as well as that of T) implies $\tilde{T}$ is right continuous on $\tilde{X}$. Set $\tilde{D} = \{[x] | x \in D\}$. We claim $\tilde{T}$ is continuous on $\tilde{D}^c$. To see this fix $[x]$ such that $[x] \cap D = \phi$, and suppose $[x_k] \to [x]$ from the left. If we write $[x] = [a_x, w_x]$, it must be that $x_k \to z_x$ and in fact $w_{x_k} \to z_x$. As $z_x \notin D$, it follows that $Tw_{x_k} \to Tz_x$, and so $\tilde{T}[x_k] = [Tw_{x_k}] \to [Tz_x]$. If $T_x = \infty$ or if $z_x = x$, there is nothing left to prove. Otherwise, by the remark following Definition 2.3 there exists $J \in \mathscr{S}$ such that $T[x] \subseteq \bar{J}$. It follows that $[Tz_x] = [Tw_x]$, and so $\tilde{T}[x_k] \to \tilde{T}[x]$.

We claim now that $(\tilde{X}, \tilde{D}, \tilde{T})$ is a P-triple. All that is left to prove is that $\tilde{T}$ is one-to-one on $(\tilde{T}^{-1}\infty)^c$. So suppose $\tilde{T}[x] = \tilde{T}[y] \in \tilde{X}$. This means by definition that $[Tw_x] = [Tw_y]$ or by Lemma 2.5 $Tw_x = w_{Tw_x} = w_{Tw_y} = Tw_y$. As T is one-to-one on $(T^{-1}\infty)^c$, $w_x = w_y$ and $[x] = [y]$.

2.7 PROPOSITION

Let $\mathscr{S}$ be a P-system for the P-triple (X, D, T), and let $(\tilde{X}, \tilde{D}, \tilde{T})$ be as defined above. Then $(\tilde{X}, \tilde{D}, \tilde{T})$ is a P-triple, which is aperiodic if (X, D, T) is.

Proof. Suppose $\tilde{T}^n[x] = [x]$ for some $[x] \in \tilde{X}$. Iteration of Lemma 2.5 implies that $[T^n w_x] = [x]$ and $T^n w_x = w_x$. If T is aperiodic, then $n = 0$. As for $\tilde{S}$, if we write $[x] = [z_x, w_x]$, one may check that $\tilde{S}[x] = [Sz_x]$, and similar arguments show $\tilde{S}$ is aperiodic if S is.

Remark. Since $\pi^{-1}\tilde{x} = [x]$ is a closed interval for each $\tilde{x} = [x] \in \tilde{X}$, where $\pi x = [x]$, the set of $\tilde{x}$ such that $\pi^{-1}\tilde{x}$ is not a singleton is at most countable. Define $\tilde{X}'$ to be the set of $\tilde{x}$ such that (a) $\pi^{-1}\tilde{x} = \{x(\tilde{x})\}$ is a singleton, and (b) $\tilde{x} \in \bigcup_n \tilde{T}^n \tilde{D}$. $\tilde{X}'$ is a set with at most countable complement, and we claim $\tilde{X}'$ is $\tilde{T}$ invariant. For consider that if $\tilde{x} = \{x\} \in \tilde{X}'$, then $T\tilde{x} = [Tx]$ because $x = w_x$. If $[Tx] = [x_0, Tx]$, then T^{-1} is continuous on $[x_0, Tx]$ because $\tilde{T}\tilde{x} \notin \tilde{T}\tilde{D}$. Then if $[x_0, Tx] \subseteq \bar{J} \in \mathscr{S}$, we have $x \in \overline{T^{-1}J}$, meaning $[x] \neq \{x\}$, a contradiction. Thus, $T\tilde{X}' \subseteq \tilde{X}'$, and a similar argument is used for establishing $T^{-1}\tilde{X}' \subseteq \tilde{X}'$. It is clear then that $\pi^{-1}\tilde{X}'$ is invariant and that $\tilde{T}\pi = \pi T$ there.

3. Invertible P-triples

A P-triple (X, D, T) will be denoted by $\mathscr{X}$, $(\tilde{X}, \tilde{D}, \tilde{T})$ by $\tilde{\mathscr{X}}$, etc. In this section the study of an aperiodic P-triple $\mathscr{X}$ will be reduced in a canonical way to the study of an invertible aperiodic P-triple $\mathscr{X}_0$. It turns out that $\mathscr{I}(\mathscr{X}) \cong \mathscr{I}(\mathscr{X}_0)$, where $\mathscr{I}(\cdot)$ denotes finite invariant Borel probability measures on X, X_0, etc. We first insert a lemma for later reference.

3.1 Lemma. If $\mathscr{X}$ is an invertible P-triple, then $TD = SD$.

Proof. Let $\mathscr{J}$ be the partition of X determined by D, each element of $\mathscr{J}$ being taken to be left closed-right open. If $J \in \mathscr{J}$, $J = [x, x')$, then $Sx' \in SD$. As T is onto, $Sx' = Tx''$ for some $x'' \in X$. If $x'' \in I \in \mathscr{J}$, then $TI \cap TJ = \phi$, meaning x'' is the left endpoint of I. That is, $x'' \in D$, so $SD \subset TD$. The reverse inclusion is proved in the same way.

If $\mathscr{X}$ is a P-triple, define $\Omega(\mathscr{X})$ by

$$\Omega(\mathscr{X}) = \{x \in X \mid T^n x = x \text{ for some } n \in \mathbb{Z}\}.$$

Both $\Omega(\mathscr{X})$ and its complement are invariant. We shall prove in this section the following:

3.2 PROPOSITION

Let $\mathscr{X}$ be an aperiodic P-triple. There exists a unique P-system $\mathscr{S}_0$ such that $\Omega = |\mathscr{S}_0|$, where $|\mathscr{S}_0| = \bigcup_{J \in \mathscr{S}_0} J$. The quotient triple $\mathscr{X}_0$ is invertible, aperiodic; it is trivial if and only if $\Omega = X$.

Remark. Let μ be a finite invariant Borel measure for T. It is clear that $\mu(\Omega) = 0$. μ is nonatomic because T is aperiodic, and so because the quotient map $\pi: \mathscr{X} \to \mathscr{X}_0$ is one-to-one on all but a countable subset of $|\mathscr{S}|^c \cap X$, $\pi\mu$ is a T_0 invariant measure on X_0. The map $\mu \to \pi\mu$ is an affine isomorphism, and therefore $\mathscr{I}(\mathscr{X}) \cong \mathscr{I}(\mathscr{X}_0)$.

To begin with let $\mathscr{P}$ be the set of connected components of $(TX)^c \cap X = \{x \mid T^{-1}x = \infty\}$.

3.3 Lemma. If $I, J \in \mathscr{P}, m, n \in \mathbb{Z}^+$ ($=$ non-negative integers), and if $(m, I) \neq (n, J)$, then

$$T^m I \cap T^n J \cap X = \phi. \tag{3.4}$$

Proof. Suppose $x \in I$, $y \in J$, $m, n \in \mathbb{Z}^+$ are such that $T^m x = T^n y \in X$. We may suppose $m \geq n$, and so set $l = m - n \geq 0$. Then $T^n T^l x = T^n x$, and so $T^l x = y$ by (iv) of Definition 2.1. As $y \notin TX$, $l = 0$ and then $x = y$. Thus, $I \cap J \neq \phi$, and $I = J$. The lemma is proved.

Define Δ to be the set

$$\Delta = \{x \in (TX)^c \cap X \mid T^n x \in D \text{ for some } n \geq 0\}.$$

By the lemma Δ is a finite set; indeed for each $y \in D$ there is *at most* one $X \in (TX)^c$ such that $T^n x = y$ for some $n \geq 0$. Δ determines a partition $\mathscr{P}'$ of $(TX)^c \cap X$ into a

finite set of left closed-right open intervals. For each $J \in \mathscr{P}'$, T is continuous on T^n for $n \geqslant 0$. Moreover, there exists $N_J < \infty$ such that $\overline{T^n J} \cap D = \phi$ for $n \geqslant N_J$. (Fo otherwise, the fact D is finite would imply $T^n J \cap T^m J \neq \phi$ for certain $m, n \geqslant 0$, $m \neq n$

By a *string* we shall mean a finite or infinite sequence $\Lambda = \{I_1, I_2, \dots\}$ of non-empt pairwise disjoint left closed-right open intervals such that for each $n \geqslant 0$ $I_n \cup I_{n+1}$ i a left closed right open interval. Λ is a *$\mathscr{P}'$-string* if there exist $n \to k_n \geqslant 0$ and $n \to I^{(n)} \in \mathscr{P}$ such that

$$I_n = T^{k_n} I^{(n)} \quad (n \geqslant 0). \tag{3.5}$$

3.6 *Lemma. Let $\mathscr{X}$ be an aperiodic P-triple, and suppose Λ is a $\mathscr{P}'$-string such tha $k_n \geqslant N_{I^{(n)}}$ in* (3.5) *for each n. Then* $\operatorname{Card}\Lambda \leqslant \operatorname{Card}\mathscr{P}'$.

Proof. Suppose $\operatorname{Card}\Lambda > \operatorname{Card}\mathscr{P}'$. There exists then a sub-$\mathscr{P}'$-string $\Lambda' \subseteq \Lambda$ which after renumbering, has $q > 1$ elements (3.5) with $I^{(1)} = I^{(q)}$. Suppose $k_q > k_1$, and se

$$k = k_q K_1, \quad J = \bigcup_{1}^{q-1} T^{k_j} I^{(j)}, \quad J' = J \cup T^{k_q} I^{(q)}.$$

Now T^k is continuous on J' and $T^k J \cap J' = T^{k_q} I^{(q)}$, meaning $T^k J$ and J form a string In fact now one sees that $\{J, T^k, T^{2k} J, \dots\}$ is a string whose elements, by Lemma 3.3 are pairwise disjoint. It follows then that the sequence $T^{mk} J$, $m \geqslant 0$, tends to zero i length and monotonically to a point x. If the convergence is from the left (resp. right then $S^k x = x$ (resp. $T^k x = x$). Aperiodicity implies $k = 0$ and $q = 1$, a contradictior The case $k_1 > k_q$ is treated similarly. The lemma is proved.

If Λ is a $\mathscr{P}'$ string, there is a bound, independent of Λ, for the number of element such that $k_n < N_{I^{(n)}}$ in (3.5) $\left(\text{i.e. } \prod_{J \in \mathscr{P}'} N_J\right)$. This and Lemma 3.5 imply there is a uniforn bound for the number of elements in any $\mathscr{P}'$ string. Thus, if if we let $\Omega^+ = \bigcup_{J \in \mathscr{P}'} \bigcup_{n \geqslant 0} T^n J$ and if $\mathscr{S}$ is the set of connected components of Ω^+, each $I \in \mathscr{S}$ is a left closed right open interval and $\bar{I} \cap \bar{J} = \phi$ for $I \neq J \in \mathscr{S}$. Clearly, $\mathscr{S}$ satisfies (b) and (c) c Definition 2.3, and so $\mathscr{S}$ is a P-system.

Let $\tilde{\mathscr{X}}$ be the quotient of $\mathscr{X}$ by $\mathscr{S}$, and let $\pi_0 : X_1 \cup \cdots \cup X_\delta \to \tilde{X}$ be the quotient map

3.7 *Lemma.* $\tilde{T}\tilde{X} \supseteq \tilde{X}$.

Proof. Let $[x] \in \tilde{X}$, and write $[x] = [z_x, w_x]$. It must be that $w_x \in TX$, else $[x]$ is no a component of Ω^+. Write $w_x = Ty$ and note the right continuity of T implies $y = w_y$ Thus, $\tilde{T}[y] = [x]$, and the lemma is proved.

Remark. If in the above one knew already that $TX \subseteq X$, then it would follow tha $\tilde{T}\tilde{X} \subseteq \tilde{X}$, and $\tilde{\mathscr{X}}$ is an invertible P-triple. To obtain $\mathscr{X}_0$ in Proposition 3.2 we shal apply what has been done so far to the inverse of $\tilde{\mathscr{X}}$. This will yield the inverse o the desired $\mathscr{X}_0$.

Let $\tilde{\mathscr{X}}$ be as above. The argument used in the proof of Lemma 3.1 implie $\tilde{T}\tilde{D} \cap \tilde{X} = \tilde{S}\tilde{D} \cap \tilde{X}$, because $\tilde{T}\tilde{X} \supseteq \tilde{X}$, and therefore $\tilde{\mathscr{X}}^{-1} = (\tilde{X}, \tilde{T}\tilde{D} \cap \tilde{X}, \tilde{T}^{-1})$ is ar aperiodic P-triple such that $\tilde{T}^{-1}\tilde{X} \subseteq \tilde{X}$. Let $\tilde{\mathscr{S}}$ be to $\tilde{\mathscr{X}}^{-1}$ what $\mathscr{S}$ was to $\mathscr{X}$, and le $\mathscr{X}_0^{-1} = (X_0, T_0 D_0, T_0^{-1})$ be the associated quotient. By the above remark $\mathscr{X}_0^{-1}$ anc

$\mathscr{X}_0 = (X_0, D_0, T_0)$, where $D_0 = T_0^{-1}(T_0 D_0)$, are aperiodic invertible P-triples.

Let $x_0 \in X_0$ pull back to $[\tilde{x}] \subseteq \tilde{X}$ and let $[\tilde{x}]$ pull back to an interval J in X. If $[\tilde{x}] = \{\tilde{x}\}$, then $J \in \mathscr{S}$, and so $J \subseteq \Omega$. If $[\tilde{x}]$ is an interval, then but for its right endpoint $[\tilde{x}]$ is a finite union of left closed-right open intervals $\tilde{J}$ such that $\tilde{T}^n \equiv x$ on $\tilde{J}$ for some $n = n(\tilde{J}) > 0$. The pullback of each $\tilde{J}$ in X is a left closed-right open subinterval $J' \subseteq J$. Let $J'' = \{x \in J' \mid T^n x = \infty\}$. Then J'' is a finite union of left closed-right open intervals which projects onto a dense subset of $\tilde{J}$ (remark at the end of § 2). It follows that each of the complementary intervals to J'' in J' projects to a point in $\tilde{X}$, and therefore $J' \subseteq \Omega$, and then $J \subseteq \Omega$. Let $\mathscr{S}_0$ be the set of nontrivial left closed-right open intervals which arise as J above (omitting the right endpoint) together with the set of components of X which map to x either from X to $\tilde{X} \cup \{\infty\}$ or from X to $\tilde{X}$ to $X_0 \cup \{\infty\}$. Then $\mathscr{S}_0$ is the desired P-triple, and X_0 is trivial if and only if $\Omega = X$. In this case $\mathscr{S}_0$ is a finite set.

As $\mathscr{S}_0$ is just the set of components of Ω, the uniqueness of $\mathscr{S}_0$ is clear.

4. Weakly minimal sets

Let $\mathscr{X}$ be an aperiodic invertible P-triple. Define for $x \in X$

$$\mathcal{O}_T(x) = \{T^n x \mid n \in \mathbb{Z}\}$$

$$\mathcal{O}_T^{\pm}(x) = \{T^n x \mid \pm n \in \mathbb{Z}^+\}$$

and similarly $\mathcal{O}_S(x), \mathcal{O}_S^{\pm}(x)$.

4.1 DEFINITION

A set $\Omega \subseteq X$ is *weakly invariant* if for each $x \in \Omega$ $\mathcal{O}_R^{\varepsilon}(x) \subseteq \Omega$, where $(R, \varepsilon) = (T, \pm)$ or $(S, \pm)$.

4.2 Lemma. For any $x \in X$ and $(R, \varepsilon) = (T, \pm)$ or $(S, \pm)$ $\overline{\mathcal{O}_R^{\varepsilon}(x)}$ is weakly invariant.

Proof. Assume $(R, \varepsilon) = (T, +)$, the other cases being similar. If $n_k \to \infty$ and $T^{n_k} x \to y$, we may, by passing to a subsequence if necessary, assume $T^{n_k} x \to y$ from the left or right. If it is, say, from the left, the finiteness of D and left continuity of S imply $T^{n+n_k} x \to S^n y$ for all n, unless $T^{n_k} x = y$ for large k. Thus, either $\mathcal{O}_T^+(x)$ is already closed or else for each $y \in \overline{\mathcal{O}_T^+(x)}/\mathcal{O}_T^+(x)$ either $\mathcal{O}_T(y)$ or $\mathcal{O}_S(y)$ is contained in $\overline{\mathcal{O}_T^+(x)}$. The lemma is proved.

The intersection of a nested collection of weakly invariant sets is weakly invariant, and so we obtain that any closed non-empty weakly invariant set contains a *minimal* weakly invariant set, a non-empty closed weakly invariant set which is minimal with respect to these properties. If M is a "weakly minimal" set, then for each $x \in M$, there is a choice of (R, ε) such that $\mathcal{O}_R^{\varepsilon}(x) \subseteq M$, and minimally plus Lemma 4.2 imply $\overline{\mathcal{O}_R^{\varepsilon}(x)} = M$.

Remark. If $x \in M$ is such that x may be approached from the left (resp. right) in M, then the above discussion implies $\mathcal{O}_S(x) \subseteq M$ (resp. $\mathcal{O}_T(x) \subseteq M$). If $M^0 \neq \phi$, then in particular $T^n M^0 \subseteq M$, $S^n M^0 \subseteq M$ for all $n \in \mathbb{Z}$. Moreover, in this situation, any x such

that $\mathcal{O}_T(x) \subseteq M$ (resp. $\mathcal{O}_S(x) \subseteq M$) is contained in an interval $[x, x+\varepsilon]$ (resp. $[x-\varepsilon, x]$ which is contained in M. Merely choose n such that $T^m x \in M^0$ (resp. $S^n x \in M^0$) and apply T^{-m} (resp. S^{-n}), recalling its right (resp. left) continuity. In particular we have that *when $M^0 \neq \phi$, M^0 is precisely the set of x which are both right and left hand limit points of M.*

4.3 PROPOSITION

Let $\mathcal{X}$ be an invertible P-triple, and let M be a weakly minimal set. If $M^0 \neq \phi$, then M is a finite union of closed intervals. (Of course it is possible that $M = X$).

Assuming $M^0 \neq \phi$ let us write, after perhaps renumbering

$$M = \bigcup_{j=1}^{\delta} M_j,$$

where $M_j = X_j \cap M$ and $M \subseteq X_1 \cup \cdots \cup X_\delta, \delta$ as small as possible. Let $J \subseteq M$ be a component of M which is a proper subset of X_j for some j ($J \subseteq M_j$). We write $J = [a, b]$. If $Ta \in M^0$, then T^{-1} cannot be continuous at Ta because $a \notin M^0$, and so $a \in D$. Thus, if $a \notin D$, Ta is the left endpoint of some other component of M. M has a finite number of components, and therefore if we replace a by $T^n a$ for some $n \geqslant 0$, we may suppose $a \in D$, $Ta \in M^0$. Applying the same reasoning to T^{-1}, there exists a component $J' = [a', b']$ of M such that $T^{n'} a' = a$ for some $n' \geqslant 0$, $a' \in TD$ (the set of possible discontinuities for T^{-1}), and $T^{-1} a' \in M^0$.

Remark. The original point a above now lies somewhere between a' and a on the list $a', Ta', \ldots, T^n a' = a$, where $n + n'$ is now written as n.

Now apply the same reasoning to b above obtaining $b' \in SD^*$, $n' \geqslant 0$, and a new $b \in D$ with the old b between b' and b on the list $b', Sb', \ldots, S^{n'} b' = b$.

The following notion corresponds to Keane's infinite distinct orbit condition for interval exchange transformations:

4.4 DEFINITION

An invertible P-triple $\mathcal{X}$ is *primitive* if

(i) $T^m D \cap T^n D = \phi$ for $m \neq n$ unless $|m - n| = 2$

(ii) $TD \cap T^{-1} D \cap X_j$ has at most one element for each j.

It is natural to replace D by the possibly smaller set of *actual* discontinuities of T. Because T is invertible $TD = SD$ and so

(i′) $S^m D \cap S^n D = \phi$ for $m \neq n$ unless $|m - n| = 2$,

(ii′) $SD \cap S^{-1} D \cap X_j$ has at most one point.

In our earlier discussion of $J = [a, b]$, we found, with slightly different notation, $Ta' \in TD$, $a'' \in D$, $n \geqslant 0$, $0 \leqslant l \leqslant n$, such that $a'' = T^{n+1} a'$. $a = T^{l+1} a'$, and a', $Ta'' \in M^0$. Now $n = 1$ by primitivity. As $a \notin M^0$, it must be that $l = 0$ and $a = Ta'$. Similarly, $b = Sb'$ for some $b' \in D$ such that $S^2 b' \in D$. Thus, $a, b \in TD \cap T^{-1} D = SD \cap S^{-1} D$, whence $a = b$ and $[a, b] = X_j$ (it is clear no component of M can be a point).

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* Recall that because (X, D, T) is invertible we have $TD = SD$.

4.5 PROPOSITION

Let M be a weakly minimal set for a primitive, aperiodic, invertible P-triple. Then either $M = X_{i_1} \cup \cdots \cup X_{i_q}$ where $1 \leqslant i_1 < \cdots < i_q \leqslant d$, or else M is a nowhere dense perfect set.

In what follows $\mathscr{X}$ is assumed to be invertible and aperiodic, and M is a nowhere dense (necessarily perfect) weakly minimal set. The complement of M in the set of components of X it intersects is the union of a countably infinite collection $\mathscr{U}$ of open intervals $(a,b) = J, a, b \in M$. If a is such that T is continuous at $T^n a$ for all $n \geqslant 0$ (resp. T^{-1} is continuous at $T^n a$ for all $n \leqslant 0$), then $T^n_a \in M$ for all $n \geqslant 0$ (resp. $n \leqslant 0$), and moreover $T^n a$ is the left endpoint of some $J_n \in \mathscr{U}$. (Of course, $T^n b$ or $S^n b$ may not be the right endpoint.) Because D is finite and T is aperiodic there are at most a finite number of points $a \in X$ such that T is discontinuous at $T^n a$ for some $n \geqslant 0$ and T^{-1} is discontinuous at $T^{-n} a$ for some $n \geqslant 0$. It follows then that there exists $J \in \mathscr{U}$ such that $T^n J \in \mathscr{U}$ for all $n \geqslant 0$ (or all $n \leqslant 0$). By aperiodicity the intervals $T^n J$, $n \geqslant 0$ (say), are pairwise disjoint. One can see readily then:

4.6 PROPOSITION

If there exists $x \in X$ such that $\mathscr{O}_T(x)$ is dense in X, then $\mathscr{X}$ is weakly minimal.

Proof. $\mathscr{O}_T(x)$ must enter each of the intervals $T^n J$ infinitely often, say $T^\alpha x, T^\beta x \in J$. If $\alpha > \beta$, then $T^\alpha x = T^\gamma T^\beta x$, $\gamma = \alpha - \beta > 0$ and so $T^\alpha x \in T^\gamma J \cap J$, an impossibility. It follows then that any weakly minimal set is a union of components, and as $\mathscr{O}_T(x)$ is dense in X, $\mathscr{O}_T(x) \subseteq M$, and $M = X$.

4.7 PROPOSITION

Let $\mathscr{X}$ be a primitive aperiodic invertible P triple, and assume the nonwandering set of $\mathscr{X}$ is X. Then any weakly minimal set is a union of components.

Remark. Proposition 4.7 is based on the corresponding result due to Keane [5] for interval exchange transformations.

5. Invariant measures for *P*-triples

In this section we consider invertible, aperiodic P-triples $\mathscr{X}$. Set up the product space $X^{\mathbf{Z}}$ with the left shift σ, and let Y be the set

$$Y = \{\mathscr{O}_T(x) \mid x \in X\} \cup \{\mathscr{O}_S(x) \mid x \in X\}$$

Y is σ-invariant, and we claim, Y is closed. For if, say, $\mathscr{O}_T(x_k) \to \omega \in X^{\mathbf{Z}}$, we may assume $x_k \to \omega_0$ (zeroth coordinate) from one side. If $x_k = \omega_0$, then $\omega = \mathscr{O}_T(\omega_0)$ while otherwise either $\omega = \mathscr{O}_S(\omega_0)$ or $\omega = \mathscr{O}_T(\omega_0)$. If $\omega_0 \notin \bigcup_{-\infty}^{\infty} T^n D$, then $\omega = \mathscr{O}_T(\omega_0)$, and so $X' = \{\mathscr{O}_T(x) \mid x \in X\}$ has at most countable complement in Y. Also, we note that because T and S are aperiodic on X, σ is aperiodic on Y. Let $\mathscr{I}$ be the cone of finite non-zero σ-invariant Borel measures on Y. Aperiodicity, implies each $\mu \in \mathscr{I}$ is non-atomic, and therefore it follows that $\mu(Y) = \mu(X')$, $\mu \in \mathscr{I}$. The zeroth coordinate map sends X' one-to-one onto X, and therefore we have a natural affine isomorphism between $\mathscr{I}$

and the corresponding object for T (and S) on X. In particular, $\mathscr{I}$ is weak-$*$ closed with compact base $\mathscr{P} = \{\mu \in \mathscr{I} \mid \mu(X) = 1\}$. Here we identify $\mathscr{I}$ with measures on X. If $K \subseteq X$ is a weakly minimal set, the set

$$Y_K = \{\mathcal{O}_T(x) \mid \mathcal{O}_T(x) \subseteq K\} \cup \{\mathcal{O}_S(x) \mid \mathcal{O}_S(x) \subseteq K\}$$

is closed, σ-invariant and projects onto K. Y_K supports a nontrivial σ-invariant measure, indeed an ergodic one, and therefore K supports such a measure for S and T.

If $\mu \in \mathscr{I}$, let K be the closed support of μ, and let $\mathscr{S}$ be the set of intervals $[a, b)$ such that (a, b) is a component of K^c. If $I, J \in \mathscr{S}$, and if $\bar{I} \cap \bar{J} \neq \phi$, then either $I = J$ or else $I = [a, b)$, $J = [b, c)$ (say) but $\mu(a, c) > 0$. Then since $\mu(a, b) = 0 = \mu(b, c)$, it must be that $\mu(\{b\}) > 0$, contradicting the fact μ is non-atomic. Because μ is T and T^{-1} invariant, the other properties of a P-system are enjoyed by $\mathscr{S}$, and therefore there exists an invertible, aperiodic P-triple $(\tilde{X}, \tilde{D}, \tilde{T})$ and map $\pi: X \to \tilde{X} \cup \{\infty\}$ such that $\pi^{-1}\tilde{X} = X_1 \cup \cdots \cup X_\delta$ (after perhaps renumbering), $\pi D = \tilde{D}$, and $\tilde{T}\pi = \pi T$ for all but a countable number of x such that $\pi^{-1}\pi x = \{x\}$.

Notice in particular that $\pi\mu$ is $\tilde{T}$ invariant and assigns positive measure to every nonempty open set in $\tilde{X}$. Thus $\tilde{\mathscr{X}}$ cannot have a weakly minimal set which is nowhere dense. Now if μ is *ergodic*, it must be that $\pi\mu$ almost every orbit is dense in $\tilde{X}$, and then $\tilde{\mathscr{X}}$ is weakly minimal (Proposition 4.6).

Recall now that K and $\tilde{X}$ have dense invariant sets V and πV such that $\pi T = \tilde{T}$ on V and $\pi^{-1}\pi v = \{v\}$, $v \in V$. If $x \in K$, the orbit closure of πx, under $\tilde{S}$ or $\tilde{T}$, is all of $\tilde{X}$. If x is a cluster point of K from, say, the right, then $\mathcal{O}_T(x) \subseteq K$ and for all $n T^n x$ is a limit point of K from the right. Then $\pi\mathcal{O}_T(x) = \mathcal{O}_{\tilde{T}}(\pi x)$ is dense in $\tilde{X}$, and in particular $\mathcal{O}_T(x)$ has V in its orbit closure. But then $\overline{\mathcal{O}_T(x)} = K$. A similar argument using S is used for left cluster points, and so K is a weakly minimal set.

5.1 PROPOSITION

Let $\mathscr{X}$ be an invertible aperiodic P-triple. If $\mu \in \mathscr{I}$ is a nonzero ergodic invariant measure for T, the closed support of μ is a weakly minimal set.

Remark. If K is a weakly minimal set then K supports a nonzero ergodic invariant measure. Thus, to understand the ergodic invariant measures for $\mathscr{X}$, one must in particular understand the weakly minimal sets; i.e. how many are there, and what is the dimension of $\mathscr{I}(K)$ for each.

Remark. In passing from $\mathscr{X}$ to $\tilde{\mathscr{X}}$ above one required knowledge only of the closed support K of μ and not of μ itself. This being so, we shall write $\mathscr{X}(K)$ for $\tilde{\mathscr{X}}$. In particular $\mathscr{X}(K)$ is defined for any weakly minimal set.

6. The genus of an invertible *P*-triple

Let $\mathscr{X}$ be an invertible P-triple, and recall that this implies

$$TD = SD \tag{6.1}$$

Using (6.1) we define a map $\tau: D \to D$ by

$$\tau x = S^{-1} T x \tag{6.2}$$

τ is one-to-one on D, and $\tau x \neq x$ precisely when $x \in D$ is an actual discontinuity of T.

As a permutation of a finite set τ has a finite number σ_τ of cyclic sets, and we define

$$\chi(\mathscr{X}) = \sigma_\tau - |D| \tag{6.3}$$

where $|\cdot|$ stands for cardinality.

As we shall see $\chi(\mathscr{X})$ is the Euler characteristic of an associated closed orientable surface, and therefore it is natural to define the *genus*, $g(\mathscr{X})$ by $2-2g = \mathscr{X}$, or

$$g(\mathscr{X}) = 1 + \tfrac{1}{2}(|D| - \sigma_\tau) \tag{6.4}$$

6.5 PROPOSITION

If K is a weakly minimal set, then K can support at most $g(\mathscr{X}(K))$ mutually singular ergodic invariant probability measures.

In §9 we shall define a "reduced genus" $g(K) \leqslant g(\mathscr{X}(K))$ with respect to which Proposition 6.5 remains true. Moreover we have:

6.6 PROPOSITION

Let $\mathscr{K}$ be a collection of distinct weakly minimal sets for $\mathscr{X}$. Then $g(K) \geqslant 1$ for each $K \in \mathscr{K}$, and

$$\sum_{K \in \mathscr{K}} g(K) \leqslant g(\mathscr{X}). \tag{6.7}$$

A consequence of Propositions 6.5 and 6.6 as well as the fact that the ergodic elements span $\mathscr{I}(\mathscr{X})$ is:

6.8 Theorem. *Let $\mathscr{X}$ be an aperiodic invertible P-triple. Then dimension $\mathscr{I}(\mathscr{X}) \leqslant g(\mathscr{X})$.*

The results stated in this section will be proved in later sections.

7. *P*-triples and foliations of surfaces

It was pointed out to me by W. Thurston that the suspension (essentially) of an interval exchange is a flow on a closed surface. In this section we make the same observation for a P-triple $\mathscr{X}$ and then calculate the Euler characteristic of the surface by means of the structure of the singularities of the suspension flow. As predicted earlier, the result will be $\chi(\mathscr{X})$ (to be defined here also for noninvertible P-triples).

Let $\mathscr{X}$ be any P-triple, and let U be set $U = X \times [0,1]$. U may be viewed as a collection of cylinders or annuli. On U there is a natural foliation $\mathscr{G} = \{(x) \times [0,1] \mid x \in X\}$, oriented from left to right. If $x \in X$ and if $Tx \neq \infty$ (resp. $Sx \neq \infty$) we declare $(x,1) \sim (Tx,0)$ (resp. $(x,1) \sim (Sx,0)$). Let $M^2 = M^2(\mathscr{X}) = u/\sim$, and let $\varphi: X \to M^2$ be the canonical projection. M^2 is a closed surface with boundary, ∂M^2 being empty if and only if $\mathscr{X}$ is invertible. Let $\mathfrak{F} = \varphi\mathscr{G}$. Then $\mathfrak{F}$ is a foliation with

singular set $\Sigma(\mathfrak{F}) \subseteq \varphi D$. Notice that M^2 is essentially the phase space for the suspension of T.

By convention $\tau = T^{-1}S$ maps X to $X \cup \{\infty\}$. We partition D into a set E of maximal "partial orbits" $s = \{x, \tau x, \ldots, \tau^{l-1}x\}$, and we note that for each $s \in E$ there are the following alternatives

(a) $\tau^l x = x$.
(b) $\tau^l x = \infty$
 (b1) $Tx = \infty, \quad S\tau^{l-1}x = \infty,$
 (b2) $Tx = \infty, \quad S\tau^{l-1}x \neq \infty,$
 (b3) $Tx \neq \infty, \quad S\tau^{l-1}x = \infty,$
 (b4) $Tx \neq \infty, \quad S\tau^{l-1}x \neq \infty.$

We define a length, $\lambda(s)$, by

$$\lambda(s) = \begin{cases} l & \text{(a)} \quad \text{or} \quad \text{(b1)} \\ l + \frac{1}{2} & \text{(b2)} \quad \text{or} \quad \text{(b3)} \\ l + 1 & \text{(b4).} \end{cases}$$

Finally, define E_a, E_b to be the sets of s such that (a), (b) above hold respectively, and set

$$\chi(\mathscr{X}) = \sum_{s \in E_a} (1 - \lambda(s)) + \sum_{s \in E_b} (\tfrac{1}{2} - \lambda(s)). \tag{7.1}$$

If $\mathscr{X}$ is invertible then $E_a = E$, and the right side of (7.1) will be $\sigma_\tau - |D|$. Thus (7.1) extends the earlier definition of $\chi(\mathscr{X})$.

7.2 PROPOSITION

With notations as above

$$\chi(M^2) = \chi(\mathscr{X}) \tag{7.2}$$

Proof: Let $s \in E$, $s = \{x, \tau x, \ldots, \tau^{l-1}x\}$. We pick $2l$ semidiscs in U about the points $(x, 1)(Sx, 0)(\tau x, 1), \ldots, (\tau^{l-1}x, 1)$ which join together in M^2 to comprise a perhaps partial neighbourhood of $\varphi(x, 1) \in M^2$. This partial neighbourhood contains $2l$ separatrices of $\varphi(x, 1)$. In case (a) above the neighbourhood is a full neighbourhood and $\varphi(x, 1) \notin \partial M^2$, while in case (b1) $\varphi(x, 1) \in \partial M^2$ and the neighbourhood is complete. In the remaining cases the neighbourhood must be enlarged to become a full neighbourhood; the full neighbourhood has $2l + 1$ separatrices in cases (b2) or (b3) and $2l + 2$ in case (b4). In all cases there are $2\lambda(s)$ separatrices, and so the index of $\mathfrak{F}$ is the sum on the right in (7.1). It follows then that $\chi(\mathscr{X}) = \chi(M^2)$. (For example, if $2M^2$ is the double of M^2 when $\partial M^2 \neq \phi$ and $2\mathfrak{F}$ the natural double of $\mathfrak{F}$, each $\varphi(x, 1) \in \partial M^2$, $x \in D$, becomes a singularity with $4\lambda(s)$ separatrices. By the index formula for a compact M^2

$$\begin{aligned} 2\mathscr{X}(M^2) &= \mathscr{X}(2M^2) \\ &= 2 \sum_{s \in E_a} (1 - \lambda(s)) + \sum_{s \in E_b} (1 - 2\lambda(s)). \end{aligned} \tag{7.3}$$

8. The inequality $b(^{\mathscr{X}}/\mathscr{S}) \leqslant b(\mathscr{X})$

Let $\mathscr{X}$ be an aperiodic P-triple. Having established that $\mathscr{X}$ and $M^2(\mathscr{X})$ have the same Euler characteristic, we turn to the problem of computing, or at least estimating, $b(\mathscr{X}) = b(M^2(\mathscr{X}))$, the first Betti number of $M^2(\mathscr{X})$. We shall prove that if $\mathscr{S}$ is a P-system for $\mathscr{X}$ then $b(^{\mathscr{X}}/\mathscr{S}) \leqslant b(\mathscr{X})$.

I am grateful to J Hempel for a fruitful discussion of intersection numbers and for particularly showing me how the matrix L^π in the interval exchange example below may be viewed as an intersection matrix.

We assume given a P-system $\mathscr{S}$ for $\mathscr{X}$, and as in earlier sections, we write $\mathscr{X}_0 = {}^{\mathscr{X}}/\mathscr{S}$ and $\pi: X_1 \cup \cdots \cup X_\delta \to X_0$, $\delta \leqslant d$.

Let $A_i = X_i \times [0,1]$. It is best here to view A_i as an annulus in the plane and $\mathscr{G}|_{A_i}$ as the radial foliation connecting the inner boundary circle c_i to the outer bounding circle C_i. Let $c = \bigcup_1^\delta c_i, C = \bigcup_1^\delta C_i$. Note that if $\pi(x,s) = (\pi x, s)$, then π maps $\bigcup_1^\delta A_i$ continuously onto $U_0 = X_0 \times [0,1]$ and the corresponding portion of $\mathscr{G}$ onto $\mathscr{G}_0 = \{(x_0) \times [0,1] \mid x_0 \in X_0\}$.

Let $\mathscr{Q}$ be the partition of $\bigcup_1^\delta \partial A_i$ defined on outer circles by D^c and on inner circles by $(TD \cup SD)^c$. Let $\mathscr{Q}_0$ be the set of $J \in \mathscr{Q}$ such that π is not constant on J; i.e. πJ is an element of the corresponding partition of ∂U_0. If $TJ = I$, $J \in \mathscr{Q}_0$, then $I \in \mathscr{Q}_0$ and $T_0 \pi J = \pi I$ (modulo interpretation at the endpoints.)

Suppose now $I, J \in \mathscr{Q}_0$, $TI = J$ and $I \subseteq A_i$, $J \subseteq A_j$ with $i \neq j$. We may glue A_i and A_j together along $\bar{I}$ and $\bar{J}$. If $I \neq C_i$ or $J \neq c_j$, the result is a disc Δ_2 with two holes. In the other case δ may be reduced by 1.

The fact $I, J \in Q_0$ implies π maps Δ_2 to the corresponding object for X_0, T_0, πI, πJ. By repeated gluings $\bigcup_{i=1}^{\delta} A_i$ is replaced by $\bigcup_{j=1}^{\varepsilon} \Delta_j$, where for each j, Δ_j is a disc with σ_j holes, $\sum_{j=1}^{\varepsilon} \sigma_j = \delta$, and there do not exist $I, J \in Q_0$, $TI = J$, with $I \subseteq \Delta_j$, $J \subseteq \Delta_k$, $j \neq k$. For each j, $\pi \Delta_j$ is a disc with σ_j holes and Δ_j/T_0 is a connected component of $M^2(\mathscr{X}_0)$.

If $I, TI \in Q_0$ lie in the boundary of $\Delta = \Delta_j$, j fixed, there exists $x \in I$ which is a limit point from both sides of $|\mathscr{S}|^c$. Choose γ to be a smooth Jordan curve in Δ from x to Tx. γ_I represents a cycle in $M^2(\mathscr{X})$. Let $\mathscr{L} = \{\gamma_I \mid I, TI \in Q_0\}$. Matters may be arranged so that if $\gamma, \gamma' \in \mathscr{S}, \gamma \neq \gamma'$, then $\gamma \cap \gamma'$ is a finite set $\pi\gamma \cap \pi\gamma' = \pi(\gamma \cap \gamma')$, and the oriented intersection number at $x \in \gamma \cap \gamma'$ is the same as that at $\pi x \in \pi\gamma \cap \pi\gamma'$. (To effect the latter, perturb γ, γ' near an intersection point x so that x becomes a point on a leaf of $\mathscr{G}$ of the form $(y) \times [0,1]$, where y is both a left and right limit of $|\mathscr{S}|^c$.) Let L be the intersection matrix

$$L_{\gamma\gamma'} = \langle \gamma, \gamma' \rangle, \tag{8.1}$$

where $\langle \gamma, \gamma' \rangle$ is the intersection number of the cycles γ, γ' in $M^2(\mathscr{X})$. (see Weyl [9] for a discussion.) Let L_0 be the corresponding matrix for $\pi\mathscr{L} = \{\pi\gamma \mid \gamma \in \mathscr{L}\}$. By the above $L_0 = L$.

Let γ be a simple closed curve in Δ/T_0. If necessary, γ may be perturbed to a

homologous curve, also denoted γ, which in Δ_0 is a one chain $\gamma = \sum_{j=1}^{p} \gamma_j$ such that for each j, $\gamma_j = \{a_j, b_j\}$, $a_j \in \gamma_{l_j} \in \pi\mathscr{L}$, $b_j \in \gamma_{r_j} \in \pi\mathscr{L}$ and $b_j \sim a_{j+1}$ modulo $T_0(b_p \sim a_1)$. Choosing $\varepsilon_j = \pm 1$ appropriately

$$\sum_{j=1}^{p-1} (\gamma_j + \varepsilon_j \gamma_{r_{j+1}}) + (\gamma_p + \varepsilon_p \gamma_{r_1})$$

is a planar cycle in Δ. If b is the first Betti number of Δ_0/T_0, it follows then that

$$b = \sigma + \operatorname{Rank} L \tag{8.2}$$

where σ is the number of holes in Δ.

If $1 \leqslant j \leqslant k \leqslant \varepsilon$, the sets $\mathscr{L}_j, \mathscr{S}_k$ associated to Δ_j, Δ_k satisfy $\langle \mathscr{L}_j, \mathscr{L}_k \rangle = 0$. That is the intersection matrix for $\mathscr{L}_j \cup \mathscr{L}_k$ has the form

$$L^{(j,k)} = \left(\begin{array}{c|c} L_j & 0 \\ \hline 0 & L_k \end{array} \right). \tag{8.3}$$

It follows then that

$$b(M^2(\mathscr{X})) \geqslant \delta + \sum_{j=1}^{\varepsilon} \operatorname{Rank}(L_j). \tag{8.4}$$

We have proved:

8.5 PROPOSITION

Let $\mathscr{S}$ be a P-system for an aperiodic P-triple $\mathscr{X}$. Then

$$b({}^{\mathscr{X}}/\mathscr{S}) \leqslant b(\mathscr{X}). \tag{8.6}$$

Example. Let $m > 1$, $\lambda = (\lambda_1, \ldots, \lambda_m)$, $\lambda_j > 0$, $\pi \in \mathfrak{S}_m$. Define $\beta_0(\lambda) = 0$ and $\beta_i(\lambda) = \lambda_1 + \cdots + \lambda_i$, $i \geqslant 1$. Let $I_i(\lambda) = [\beta_{i-1}(\lambda), \beta_i(\lambda))$. Define λ^π by $\lambda_i^\pi = \lambda_{\pi_i^{-1}}$, and let $T: [0, |\lambda|) \to [0, |\lambda|)$, $|\lambda| = \beta_m(\lambda)$, be linear on $I_i(\lambda)$ slope 1 with $TI_i = I_{\pi i}(\lambda^\pi)$. T is the (λ, π) *interval exchange.* Define an $m \times m$ matrix L^π by

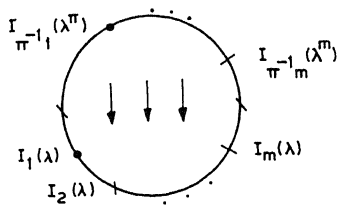
$$L_{ij}^\pi = \begin{cases} 1 & i < j, \quad \pi i > \pi j \\ -1 & i > j, \quad \pi i < \pi j \\ 0 & \text{otherwise.} \end{cases}$$

Then as was noted in [7], [8]

$$Tx = x + (L^\pi \lambda)_i \quad (x \in I_i(\lambda)).$$

It was proved in [8] that the cone of finite T-invariant measures has dimension at

most $\frac{1}{2}$Rank L^{π}. Let Δ be the disc in (8.7) with the indicated partition



(8.7)

If particles flow vertically in the negative directions, and if $I_i(\lambda)$ and $I_{\pi i}(\lambda^{\pi})$ are identified by a gluing map, matters may be arranged so that the first return map to the upper semi-circle of Δ is isomorphic with T. For each j let γ_j be the line segment from $x \in I_j^0(\lambda)$ to $Tx \in I_{\pi j}(\lambda^{\pi})$. Then as was pointed out by J. Hempel, the associated intersection matrix is L^{π}.

Remark. Because $\{\gamma_j\}$ spans $H_1({}^{\Delta}/T, \mathbb{Z})$ there exists $A \in SL(m, \mathbb{Z})$ such that

$$A^t L^{\pi} A = \left(\begin{array}{cc|c} 0 & I & \\ -I & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right) \tag{8.8}$$

where I is the $a \times a$ identity matrix, $2a = b(M^2({}^{\Delta}/T))$. The existence of such an A was proved by other means in [8]. The consequence of (8.8), that $\operatorname{Det} L^{\pi} = 0$ or 1, was noted in [7] and proved by still other means in [8].

It will be useful for later reference to consider in slightly greater detail the case $\mathscr{X}_0 = {}^{\mathscr{X}}/\mathscr{S}$ which arises in the canonical reduction of $\mathscr{X}$ to an invertible P-triple. We shall prove:

8.9 PROPOSITION

Let $\mathscr{X}_0 = {}^{\mathscr{X}}/\mathscr{S}$ be as in Section 3, and let ρ be the number of boundary components of $M^2(\mathscr{X})$. Then

$$b(\mathscr{X}_0) + \rho \leqslant b(\mathscr{X}). \tag{8.10}$$

Begin with $U = X \times [0, 1]$, and define $E = \bigcup_{J \in \mathscr{S}} \bar{J} \times [0, 1]$. Let $\mathscr{G}^{\perp}$ be the foliation of U into circles $X_i \times (s)$. For each $L \in \mathscr{G}^{\perp}$ $L \cap E$ is a union of pairwise disjoint closed intervals, called components of E. The gluing map $\sigma: U \to M^2(\mathscr{X})$ is a homeomorphism on $X \times (0, 1)$ and the images of components in this set under σ will be called components.

If $I \subseteq \partial U$ is a component of E, and if there exists $J \in \mathscr{Q}_0$ (see the beginning of this section) such that $I \subseteq J$, then again we call σI a component. For in this case the only

component I' of E such that $\sigma I' \cap \sigma I \neq \phi$ is for a choice of ± 1 $T^{\pm 1}I \subseteq T^{\pm 1}J$. See the remark following Definition 2.3.

The components which have so far not been accounted for are finite in number, lie in ∂U, and project under σ to a graph G in $M^2(\mathscr{X})$.

We may suppose $M^2(\mathscr{X})$ is connected and so, being oriented, is a sphere with g handles and ρ holes. Thus,

$$b(\mathscr{X}) = 2g + \rho. \tag{8.11}$$

Let $M_0^2 = G^c$, and consider M_0^2 as a surface with boundary. We write M_0^2 as a union of connected components,

$$M_0^2 = \bigcup_{j=1}^{l} M_j^2,$$

where for each $j M_j^2$ is a sphere with g_j handles and ρ_j holes. M_0^2 has no more handles than $M^2(\mathscr{X})$, and therefore

$$\sum_{j=1}^{l} g_j \leqslant g. \tag{8.12}$$

We introduce an equivalence relation $\sim$ on M_0^2, saying $x \sim y$ if either x and y lie in the same component of ∂M_0^2 or if x and y lie in the same component of σE. Let N_0^2 be the quotient by $\sim$, and let $\varphi: M_0^2 \to N_0^2$ be the quotient map. As $\sim$ is (clearly) a closed relation, N_0^2 is T_2.

Let $\mathfrak{F}^\perp$ be the foliation of $M^2(\mathscr{X})$ coming from $\mathscr{G}^\perp$, and let $\mathfrak{F}_0^\perp$ be the corresponding foliation of M_0^2. $\mathfrak{F}^\perp$ and $\mathfrak{F}_0^\perp$ consist of closed leaves and connections between singularities. Because $(M^2(\mathscr{X}), \mathfrak{F}(\mathscr{X}))$ arise from a P-triple, each singularity has only hyperbolic sectors.

$x \in L \in \mathfrak{F}_0^\perp, L$ a closed leaf, then either L is a component of σE or it is not. In the first case N_0^2 is a line segment in a neighbourhood of φx. In the second case N_0^2 is a disc near φx. Now we must consider φ on ∂M_0^2.

Let V be a boundary component of M_0^2. Then φV is a point in N_0^2, and we claim a neighbourhood of φV is either a half closed line segment or else a disc. To see this note that if $a \in V$ is a singularity of $\mathfrak{F}_0$, and if W is a hyperbolic sector at a, then $\mathfrak{F}_0^\perp$ has a leaf L which has $\bar{L}/L \cap W = \{a\}$, at least if W is small. Moreover, L must correspond to an element of $\mathscr{Q}_0$, else V would not be a component of ∂M_0^2. It follows then that if we take W large enough that φ is not constant on $L \cap W$, then φW is a sector at φa.

Now choose an annular neighbourhood V_0 of V so that each hyperbolic sector at each $a \in \Sigma \cap V$ is mapped by φ to a wedge. Then if $\Sigma \cap V \neq \phi, \varphi V_0$ is a disc about φV. If $\Sigma \cap V = \phi$, and if V_0 is small, then φV_0 is a line segment with φV at one end. Note that in all cases φV is a point in N_0^2 and in particular $\varphi^*[V] = 0$ in $H_1(N_0^2, \mathbb{Z})$. (N_0^2 is a compact surface with certain line segments attached, and so $\pi_1(N_0^2), H_1(N_0^2, \mathbb{Z})$, etc. are meaningful.)

A moment's reflection shows that the cycles chosen in the proof of Proposition 8.5 to map onto a spanning set for $H_1(M^2(\mathscr{X}/\mathscr{S}), \mathbb{Z})$ are cycles in $G^c \subseteq M_0^2$. Thus, $\varphi^*: H_1(M_0^2, \mathbb{Z}) \to H_1(N_0^2, \mathbb{Z})$ is surjective. In addition $\varphi^*[V] = 0$ for every boundary

component, and so the rank of φ^* is at most $2\sum_{j=1}^{l} g_j \leqslant 2g$. It follows that $b(N_0^c)+\rho \leqslant b(\mathscr{X})$, where ρ is as in (8.10).

Let the numbering be chosen so that $\pi: X_1 \cup \cdots \cup X_\delta \to X_0$ (onto). Let $W \subseteq M_0^2$ be the union of the open cylinders $X_k \times (0,1)$ together with boundary arcs which are elements of $\mathscr{Q}_0$ (and are glued together in pairs). φ maps $\bar{W}$ onto N_0^2 minus the extra line segments, and in fact φW is a closed surface. In fact, in considering how the gluing map T_0 is defined, φW and $M^2(\mathscr{X}_0)$ are the same.

9. Foliations and *P*-triples

Let $(M^2, \mathfrak{F})$ be as in theorem 1. We shall replace this pair with a pair of the same description, although possibly smaller genus, and with isomorphic cones of invariant measures. The new foliation will have no closed leaves, and for every separatrix L of a hyperbolic sector, $\bar{L}/L$ will be infinite.

Let G be the union of the set of closed leaves and separatrices L of hyperbolic sectors which do *not* have the property mentioned in the last paragraph. G is a closed set (if we add certain elements of $\Sigma(\mathfrak{F})$ to it). We view G^c as a foliated surface with boundary M_0^2, where certain singularities may exist on the boundary. If H is a component of ∂M_0^2 other than a closed leaf, then H arises from a sequence of separatrices $L_1, L_2, \ldots, L_q$ joined end to end. In M^2 it may happen that $L_i = L_j$ for some $i \neq j$, but L_i and L_j are "pulled apart" in M_0^2.

Now identify each boundary component H to a point $[H]$. In the case of a closed leaf $[H]$ becomes an attractor (a focus). In the case above, $[H]$ is a singularity with parabolic and/or hyperbolic sectors.

Let M_1^2 be the surface which results from the above identifications, and let $\mathfrak{F}_1$ be the corresponding foliation. To make $\mathfrak{F}_1\, C^1$ use the fact $\mathfrak{F}$ is oriented to find a C^0 vector field X on $M_1^2/\Sigma(\mathfrak{F}_1)$ having leaves of $\mathfrak{F}_1$ for its orbits. Then multiply X by a C^1 function F which vanishes rapidly on $\Sigma(\mathfrak{F}_1)$, and only there.

Now $\mathfrak{F}_1^2$ has no closed leaf, and if $L \in \mathfrak{F}_1^2$ is a separatrix of a hyperbolic sector of some element of $\Sigma(\mathfrak{F}_1)$, then $\bar{L}/L$ is infinite. Finally, M_1^2 is a sphere with no more handles than M^2 had, and so genus $(M_1^2) \leqslant$ genus(M^2).

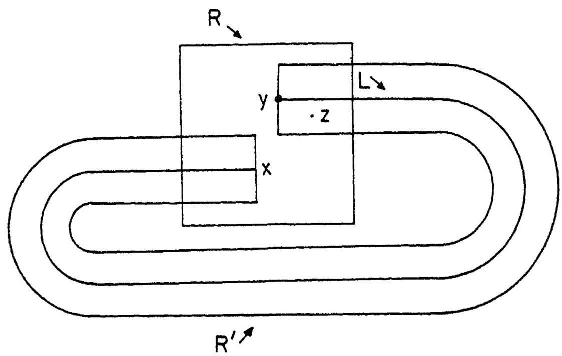
Now $(G^c, \mathfrak{F})$ and $(N^c, \mathfrak{F}_1)$ are isomorphic, where N is a subset of $\Sigma(\mathfrak{F}_1)$, $N = \{[H] \mid H$ a component of $\partial M_0^2\}$. A nonatomic foliation invariant measure μ involves the assignment of a Borel measure to each transversal in such a way that the "first arrival map" from a transversal C_1 to a transversal C_2 is as measure preserving as possible. If all the measures assigned are nonatomic, we say μ is nonatomic. If μ is nonatomic, then μ is well defined on both $(G^c, \mathfrak{F})$ and $(N^c, \mathfrak{F}_1)$. It follows that $\mathscr{I}(\mathfrak{F}) \cong \mathscr{I}(\mathfrak{F}_1)$.

The following elementary but useful lemma is due to Aranson:

9.1 Lemma. (*Aranson* [0]). *Let $\mathfrak{F}$ be a C^1 foliation of M^2. If $L \in \mathfrak{F}$, and if R is a flow box such that $L \cap R$ is not connected, then there exists a closed transversal C such that $C \cap L \neq \phi$.*

Proof. (Oral communication from W Thurston). There is a segment of L where

endpoints x, y lie in R and such that $L \cap R$ has two pieces on different "levels" of R (see 9.2). Construct a thin flowbox R' so that the segment of interest lies in $R \cup R'$. Connect x to z



(9.2)

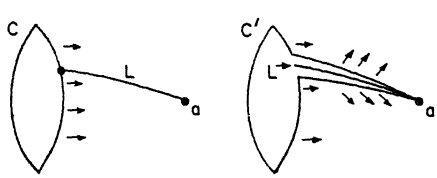
by a line segment in R (which is transverse automatically), and the n connect z to x by a line segment in R'. The union of these segments is a closed transversal C.

Now let $(M^2, \mathfrak{F})$ be as in theorem 1. If $G \neq \phi$, where G is as at the beginning of this section, reduce the pair so that it becomes empty. As $\mathfrak{F}$ is assumed to be oriented we may, for each $\sigma \in \Sigma$, speak of the set of "inward" separatrices of hyperbolic sectors at σ. Let R be the totality of such separatrices as σ varies in Σ. For each $L \in R$ choose by Aranson's lemma a closed transversal C_L intersecting L. Matters may be arranged so that $C_L \cap C_{L'}$, $L \neq L'$, is a finite set, and each intersection is transverse. By a device of Thurston's one replaces the set $\{C_L | L \in R\}$ by a set of closed transversals with the same saturation but no intersections. At each point of $C_L \cap C_{L'}$ modify as indicated in the picture:

Before After

After a finite number of such steps we obtain a set $\mathscr{R}$ of pairwise disjoint closed transversals.

If $L \in R$, there exists $C \in \mathscr{R}$ and a point $x \in L \cap C$ such that the "forward component" of L beyond x intersects no other element of $\mathscr{R}$. We assume, by discarding a subset of $\mathscr{R}$ if necessary, that each $C \in \mathscr{R}$ plays this role for at least one $L \in R$. Now alter $C \in \mathscr{R}$ at each $x \in L \cap C$ as above so that C remains closed and transverse but contains the singularity associated to L. (Thus, C is not really transverse everywhere, but the orientation of its transversality, where defined, does not change. See below.)



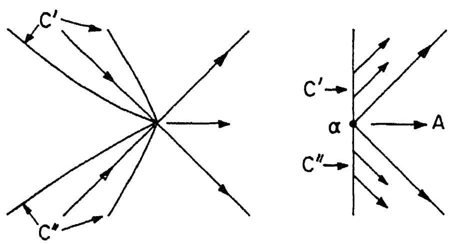
Before After

To make C transverse (except at a) iterate the argument in Aranson's lemma choosing longer and thinner flow boxes while progressing on L toward a.

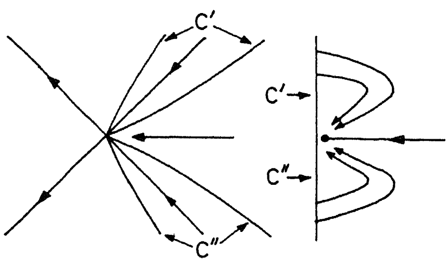
Call the new set of curves $\mathscr{R}'$, and let $\Lambda = \left(\bigcup_{C' \in \mathscr{R}'} C \right)^c$. If $\mathscr{R}' = \phi$, then $\Lambda = M^2$, and M^2 is the sphere on the torus. In the latter case $\mathfrak{F}$ is without singularities.

We regard Λ as a foliated surface with boundary, there possibly being singularities on the boundary. We may arrange so that a corresponding vector field is continuous up to the boundary.

True singularities occur on $\partial\Lambda$ only when a parabolic sector has come between two hyperbolic sectors, as in (9.3)–(9.4). The right hand diagrams



(9.3)



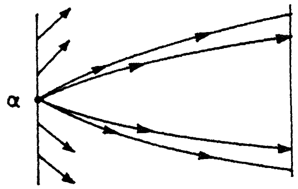
(9.4)

show the relevant neighbourhood after a C^0 change of variable. Note that the double of the right hand diagram (9.3) has two hyperbolic and two parabolic sectors and therefore index 0. The double of the right hand diagram in (9.4) has two elliptic sectors and hence index 2.

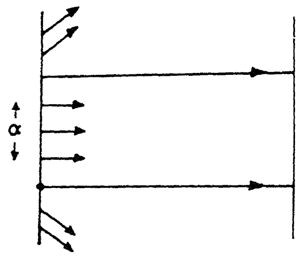
If Λ' is a connected component of Λ, then $\Sigma \cap \text{Interior } \Lambda'$ is empty or consists of attractors. Thus, Λ' is M^2, a disc, or a cylinder. If Λ' is a disc, then any leaf L which intersects the interior of Λ' has an infinite component contained in Λ'. If $\Lambda \neq M^2$, then $\mathfrak{F}$ has hyperbolic sectors, and it must be that at least one component of Λ is a cylinder.

Let $\Lambda_1, \ldots, \Lambda_d$ be the cylindrical components of Λ, and assume $d \geqslant 1$. We assume $\mathfrak{F}$ is oriented away from the left boundary λ_j of Λ_j toward the right boundary ρ_j. (By our construction, ρ_j is an element of $\mathscr{R}'$ for each j). Let $\lambda = \bigcup_{j=1}^{d} \lambda_j$, and let $T: \lambda \to \lambda$ be the Poincaré map. T is one-to-one, continuous, and locally orientation preserving on a maximal open subset Ω of λ. Ω is bounded by points on separatrices of hyperbolic sectors, and therefore Ω is a finite union of arcs. Define $T \equiv \infty$ on the interior of $\lambda \cap \Omega^c$, and then arrange that T be right continuous. If D_λ is the set of discontinuities of T, then $\mathscr{L} = (\lambda, D_\lambda, T)$ is a P-triple.

For each j, ρ_j has no singularities and so by the Poincaré Bendison theorem (for an annulus) the parabolic sector A in (9.3) reaches ρ_j as pictured below



We pull α apart so that the picture looks like



(9.5)

α cannot lie in TD^c_Λ, and so no new discontinuities of T are created. $T^{-1} \equiv \infty$ on the arc α in (9.5), and so T^{-1} may have one more discontinuity. (That is, $TD_\lambda \cup SD_\lambda$ may increase by one.)

Now $(\Lambda, \mathfrak{F})$ is C^0 (C^1 away from $\partial\Lambda$) equivalent with $M^2(\mathscr{L})$, $\mathfrak{F}(\mathscr{L})$, and therefore $\mathscr{I}(\mathfrak{F}) \cong \mathscr{I}(\mathfrak{F}(\mathscr{L}))$. $M^2(\mathscr{L})$ is a sphere with g handles and ρ holes, and by the construction $g \leqslant \text{genus}\, M^2$. It follows then that the canonical reduction, $\mathscr{X}$, of $\mathscr{L}$, to an invertible P-triple has $g(\mathscr{X}) \leqslant \text{genus}\, M^2$ (Proposition 8.9).

If M is a quasiminimal set for $\mathfrak{F}$, then certainly $M \subseteq \bigcup_{j=1}^{d} \Lambda_j$, and $M \cap \lambda = K$ is a weakly minimal set. By the same token, any weakly minimal set for $\mathscr{L}$ gives rise to a quasiminimal set for $\mathfrak{F}$. The map $\mathscr{L} \to \mathscr{X}$ sends distinct weakly minimal sets to distinct weakly minimal sets, and so we must finally estimate the number of weakly minimal sets for $\mathscr{X}$.

Let $K_1, \ldots, K_p$ be distinct weakly minimal sets, and let $\mathscr{S}_1, \ldots, \mathscr{S}_p$ be the associated P-systems such that $\mathscr{X}(K_j) = {}^{\mathscr{X}}/\mathscr{S}_j$ for each j. Reduce $M^2(\mathscr{X}(K_j))$ by removing saddle connections, as in the beginning of this section, and define $g(K_j)$ to be the genus of the reduction. Of course the reduction has an isomorphic cone of invariant measures.

Let $\mathscr{S}$ be the P-system

$$\mathscr{S} = \{I_1 \cap \cdots \cap I_p | I_j \in \mathscr{S}_j, 1 \leqslant j \leqslant p\}$$

and set $\mathscr{X}_0 = {}^{\mathscr{X}}/\mathscr{S}$. $\mathscr{X}_0$ has weakly minimal sets $K_1^0, \ldots, K_p^0$, each with interior. Now $M^2({}^{\mathscr{X}}/\mathscr{S})$ may have saddle connections, and if we remove them, the corresponding subsets $M_1^0, \ldots, M_p^0$ become quasiminimal with *every* leaf dense. Thus, in the new setting $M_j^0 \cap M_k^0 = \phi$, and

$$\sum_{j=1}^{p} \text{genus}\, M_j^0 \leqslant g(\mathscr{X}_0).$$

But also one sees readily that M_j^0 is $M^2(\mathscr{X}(K_j))$ as it was reduced, and so finally we obtain

$$\begin{aligned}\sum_{j=1}^{p} g(K_j) &\leqslant g(\mathscr{X}_0)\\ &\leqslant g(\mathscr{X})\\ &\leqslant \text{genus}\, M^2.\end{aligned}$$

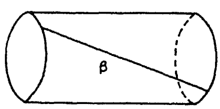
Of course, $g(K_j) \geqslant 1$ for each j.

By Katok [4] a quasiminimal foliation $(M_j^0, \mathfrak{F}^0)$ has

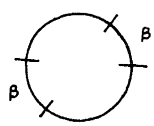
$$\dim \mathscr{I}(\mathfrak{F}^0) \leqslant \text{genus}\, M_j^0 \tag{9.6}$$

and so theorem 1 is proved.

For another approach to (9.6), let $(M^2, \mathfrak{F})$ be quasiminimal, and let C be a closed transversal to $\mathfrak{F}$ (by Aranson's lemma). The Poincaré map on C creates a P-triple $\mathfrak{C} = (C, D, T)$, and $M^2(\mathfrak{C})$ has genus no larger than that of M^2, as above. Make a transverse cut from left to right in $C \times [0, 1]$



and open up to a disc



The foliation is vertical (downwards), and T (extended to β) as a map from the upper semicircle to itself gives rise to a diagram (8.7) and in particular a permutation π. Of course we know

$$\text{genus}\, M^2(\mathfrak{C}) = \tfrac{1}{2}\, \text{Rank}\, L^\pi. \tag{9.7}$$

Each $\mu \in \mathscr{I}(\mathfrak{C})$ may be used to define arc length on the upper semicircle of (8.7), and in terms of it T becomes isomorphic to a minimal (λ, π) interval exchange. By [8] this yields an isomorphism between $\mathscr{I}(\mathfrak{C})$ and the cone of invariant measures for a fixed minimal (λ, π) interval exchange. That cone has dimension at most $\frac{1}{2}\text{rank}\, L^\pi$. Thus, Katok's bound and the one from [8] are equivalent.

Remark. The set $\mathscr{R}'$ can be viewed as a collection of cycles in G^c transverse to $\mathfrak{F}$. If $L \subseteq G$ is a (closed) leaf or compact separatrix, the only obstruction to creating a cycle C transverse to $\mathfrak{F}$ and intersecting L is that either (i) L is homologous to 0 or (ii) L extends to a saddle loop which is homologous to 0. (A saddle loop is a cycle formed from compact separatrices oriented by $\mathfrak{F}$.) Therefore, there exists a set $\mathscr{R}''$ of transverse cycles with saturation $\mathfrak{F}$, provided $\mathfrak{F}$ contains no saddle loop homologous to 0. (In the case (i) above, L is homotopic to a saddle loop). If $\mathfrak{F}$ has no parabolic sectors,

then $\mathfrak{F}$ is without singularities on $\partial\Lambda_j$, $1 \leqslant j \leqslant d$, and $\bigcup_{j=1}^{d} \Lambda_j = \Lambda = \left(\bigcup_{c'' \in \mathscr{C}''} C'' \right)^c$. Finally, assume there exists $\mu \in \mathscr{I}(\mathfrak{F})$ such that $\mu(U) > 0$ for every nontrivial arc $U, U \subseteq \Lambda_j$, $1 \leqslant j \leqslant d$. If C_j is the base of Λ_j, represent Λ_j as a right circular cylinder Γ_j of radius $r_j = \mu(C_j)/2\pi$ and height (say) 1, and straighten out $\mathfrak{F}|_{\Lambda_j}$ so that it is equivalent to the vertical foliation of Γ_j. Use the developing map to realize Γ_j as a rectangle of base $\mu(C_j)$ and height 1. Now M^2 acquires a complex structure from the atlas (Λ_j, F_j) here constructed, and away from $\Sigma(\mathfrak{F}) \cap (\cup \Lambda_j)$ the coordinate transitions have the form $z \to z + c$. Moreover, this atlas endows M^2 with a natural holomorphic 1-form, and this form (and the complex structure) extends to $\Sigma(\mathfrak{F})$ to have a zero of order v at $s \in \Sigma(\mathfrak{F})$, where $2\pi(v+1)$ is the total angle of the rectangles at s.

Theorem. *Let $\mathfrak{F}$ be an oriented foliation of M^2, and suppose* (i) *$\mathfrak{F}$ has only hyperbolic singularities,* (ii) *there exists an everywhere positive $\mu \in \mathscr{I}(\mathfrak{F})$, and* (iii) *$\mathfrak{F}$ has no saddle loop homologous to zero. There exists a complex atlas U on M^2 and a U-holomorphic 1-form ω such that $\mathfrak{F}$ is homeomorphic to the foliation tangent to ker* $\operatorname{Re}\omega$.

References

[0] Aranson S H, The absence of nonclosed Poisson-stable semi-trajectories and trajectories doubly asymptotic for a double limit cycle for dynamical systems of the first degree of structural instability on orientable two-dimensional manifolds, *Math. Sb.* **76** (1968) 214–230 = *Math. USSR Sb.* (1968) 205–220

[1] Bebutoff M and Stepanoff W, Sur la mesure invariente dans les systèmes dynamiques qui ne different que par le temps, *Recueil Math.* (*Mat. Sbornik*) **7** (1940) 143–166

[2] Cherry T M, Analytic quasi-periodic curves of discontinuous type on a torus, *Proc. London Math. Soc.* **44** (1937) 175–215

[3] Hartman P, *Ordinary different equations* (New York: Wiley and Sons) (1964) (1973)

[4] Katok A B, Invariant measures of flows on oriented surfaces, *Sov. Math. Dokl.* **14** (1973) 1104–1108

[5] Keane A, Interval exchange transformations, *Math. Z.* **141** (1975) 77–102

[6] Maier A G, Trajectories on the closed orientable surfaces, *Math. Sb.* **12** (1943) 71–84 (Russian)

[7] Rauzy G, Echanges d'Intervalles et Transformations Induites, *Acta Arith.* **34** (1978) 315–328

[8] Veech W A, Interval exchange transformations, *J. d'Anal. Math.* **33** (1978) 222–278

[9] Weyl H, *The concept of a Riemann surface* (translated by G R MacLane) (Reading: Addison-Wesley Publ. Co.) (1964)

ɔc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 1, April 1989, pp. 49–73.
Printed in India.

ɔsitive solutions of the semilinear Dirichlet problem with critical owth in the unit disc in $\mathbb{R}^2$

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Abstract. We prove the existence of a positive solution of the following problem

$$-\Delta u = f(r, u) \quad \text{in } D$$
$$u > 0$$
$$u = 0, \quad \text{on } \partial D$$

where D is the unit disc in $\mathbb{R}^2$ and f is a superlinear function with critical growth.

Keywords. Sub-critical growth, critical growth, super critical growth; Laplacian; Palais-Smale condition; Semilinear Dirichlet problem; unit disc.

Introduction

D be the unit disc in $\mathbb{R}^2$. We are looking for positive radial solutions of the owing problem: Find u in $C^2(D) \cap C^0(\bar{D})$ such that

$$\begin{aligned} -\Delta u &= f(r, u) \quad \text{in } D \\ u &> 0 \\ u &= 0, \quad \text{on } \partial D \end{aligned} \tag{1.1}$$

ere f is superlinear, $f(r, 0) = 0$, $(\partial f/\partial t)(r, 0) < \lambda_1$ with λ_1 being the first eigenvalue he Dirichlet problem. For $n \geqslant 3$ and f of critical growth, Brezis–Nirenberg [4] died the existence and non-existence of solutions of problem (1·1). For $n = 2$, the ical growth is of exponential type whereas in the case of $n \geqslant 3$, it is of polynomial e and the method adopted for $n \geqslant 3$ fails in the case of $n = 2$.

Carleson–Chang [5] obtained a positive solution for $f(u) = \lambda u \exp(\lambda u^2)$ with $\lambda < \lambda_1$ via a variational method. For growths of type $f(u) = u^m \exp(bu^2)$, Atkinson–etier [3] used the shooting argument to obtain a solution of (1.1). They assumed t $\log f$ is strictly convex for large u.

n this paper we relax the conditions on f and use a variational method to obtain olution of (1.1). Since we are interested in radial solutions, (1.1) is equivalent to

finding an u in $C^2(D)\cap C^0(\bar{D})$ with u radial and satisfying

$$\begin{aligned} L_1 u \equiv -(ru')' = f(r,u)r \quad &\text{in } [0,1) \\ u>0 \quad &\text{in } [0,1) \\ u'(0)=u(1)=0. \end{aligned} \tag{1.2}$$

where $u' = \mathrm{d}u/\mathrm{d}r$.

The idea of the method is to approximate the energy functional by functionals satisfying Palais–Smale conditions. Then obtain the critical points of these approximate functionals by a constrained minimization problem similar to that of Zeev–Nehari [8] and then pass to the limit. The method of the proof is in the spirit of Brezis–Nirenberg [4]. Here, we also get a constant "a" which is strictly less than the best possible constant and thereby the existence of solutions of (1.2) is guaranteed.

In [1] we also prove the existence of infinitely many solutions of (1.1) when f is odd and of critical growth. Also in [2] we prove the existence of solutions of (1.1) if D is replaced by an arbitrary smooth domain.

2. Statements

Let $E=\{u\in C^1[0,1]; u(1)=0\}$. For $0\leqslant\alpha\leqslant 1$ and u in E define

$$|u|_\alpha^2 = \int_0^1 u^2(r) r^\alpha \,\mathrm{d}r$$

$$\|u\|_\alpha^2 = \int_0^1 u'(r)^2 r^\alpha \,\mathrm{d}r.$$

Let H_0^α be the completion of E with respect to $\|\cdot\|_\alpha$. Define the operator L_α by

$$L_\alpha = -\frac{1}{r^\alpha}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^\alpha \frac{\mathrm{d}}{\mathrm{d}r}\right). \tag{2.1}$$

Let $(\lambda_\alpha, \phi_\alpha)$ be the first eigenvalue and the corresponding first eigenvector with $\phi_\alpha(0)=1$ of the following eigenvalue problem.

$$\begin{aligned} L_\alpha\phi = \lambda\phi \quad &\text{in } [0,1] \\ \phi'(0)=\phi(1)=0. \end{aligned} \tag{2.2}$$

DEFINITION 2.1

Let $f:[0,1]\times[0,\infty)\to[0,\infty)$ be a C^1-function. We say f is of class **A** if

(i) $f(r,0)=0$.
(ii) There exists a $\delta_0>0$ and for $(r,t)\in Q_{\delta_0}\equiv[0,\delta_0]\times[0,\infty)(\partial f/\partial r)(r,t)\geqslant 0$.
(iii) There exists a $t_0>0$ such that $f(r,t)<\lambda_1 t$ for all $(r,t)\in[0,1]\times[0,t_0]$.
(iv) There exist constants $t_1>0$, $\beta>2$ such that $\beta F(r,t)\leqslant f(r,t)t$ for all $(r,t)\in[0,1]\times[t_1,\infty)$ where $F(r,t)=\int_0^t f(r,s)\,\mathrm{d}s$.

Let

$$A' = \left\{ f \in A; \frac{\partial f}{\partial t} > \frac{f}{t} \text{ in } [0,1] \times (0,\infty) \right\}.$$

We consider the following three types of functions in our discussions.

Sub-critical: f in A is said to be sub-critical if there exists a $\delta > 0$ and for every $\varepsilon > 0$

$$\sup_{(r,t)\in[0,\delta]\times[0,\infty)} f(r,t)\exp(-\varepsilon t^2) < \infty \tag{2.3}$$

Critical: f in A' is said to be critical if there exists $\delta_1 > 0$ such that

(i) $f(r,t) = h(r,t)\exp(b(r)t^2) \quad \forall (r,t)\in Q_{\delta_1} \equiv [0,\delta_1]\times[0,\infty)$
(ii) $\forall \varepsilon > 0$,

$$\sup_{(r,t)\in Q_{\delta_1}} h(r,t)\exp(-\varepsilon t^2) < \infty \tag{2.4}$$

(iii) For every $\varepsilon > 0$, $h(0,t)\exp(\varepsilon t^2) \to \infty$ as $t \to \infty$.

Super critical: $f \in A'$ is said to be super critical if for every $c > 0$

$$\sup_{\|w\|=1} \int_0^1 f(r,cw)wr\,\mathrm{d}r = \infty. \tag{2.5}$$

For $f \in A$, $0 \le \alpha \le 1$, let $\sum_\alpha$ be the set of C^2-solutions of the following problem

$$\begin{aligned} L_\alpha u &= f(r,u) \quad \text{in } [0,1] \\ u &> 0 \\ u'(0) &= u(1) = 0. \end{aligned} \tag{2.6}$$

DEFINITION 2.2

u in $H_0^1(D)$ is said to be a weak solution of (1.2) if

(i) $u > 0$ in $[0,1)$

(ii) $\displaystyle\int_0^1 f(r,u)ur\,\mathrm{d}r < \infty$ (2.7)

(iii) $\forall \phi \in C^2[0,1]$ with $\phi(1) = 0$

$$\int_0^1 u(L_1\phi)r\,\mathrm{d}r = \int_0^1 f(r,u)\phi r\,\mathrm{d}r.$$

Since we are interested in only positive solutions of (1.2) and hence extending f for $t \leqslant 0$ is irrelevent. Therefore we make the following conventions.

1) Whenever we say f is in A, then we extend f by $f(r,t) = 0$ for $t \leqslant 0$ and $r \in [0,1]$.
2) Whenever we say f is in A', then we extend f by $f(r,t) = -f(r,-t)$ for $t \leqslant 0$. (2.8)

For u in H_0^α, define

$$\bar{I}_\alpha(u) = \tfrac{1}{2}\|u\|_\alpha^2 - \int_0^1 F(r,u)r^\alpha \,\mathrm{d}r. \tag{2.9}$$

$$l_\alpha = \inf_{\Sigma_\alpha} \bar{I}_\alpha.$$

Then we have

Theorem 2.1. *Let f be in A. Then there exists an $\alpha_0 < 1$ such that for every $\alpha_0 \leqslant \alpha < 1$, Σ_α is non-empty and $\{l_\alpha\}$ is bounded. Let $l = \underline{\lim}_{\alpha\to 1} l_\alpha$. Suppose there exists $b > 0$, $M > 0$ such that*

(i) $f(r,t) \leqslant M \exp(bt^2)$ *for all* $(r,t) \in [0,\delta] \times [0,\infty)$

(ii) $bl < 1$. (2.10)

Then there exists a solution u of (1.2).

COROLLARY 2.1

If f is sub-critical, then there exists a solution.

Proof. If f is sub-critical, we can take b as small as we want and satisfying (i) and (ii) of Theorem (2.1). Hence the solution exists.

Criterion to satisfy (2.10). Let f be in A satisfying (i) of Theorem (2.1). Suppose there exists an $m > 0$ such that

$$\int_0^{1/2} F\left(r, \frac{m}{2}\right) r \,\mathrm{d}r \geqslant 2m^2.$$

$$2m^2 b < 1 \tag{2.11}$$

Then f satisfies (ii) of Theorem (2.1).

For f in A^1 and for $0 \leqslant \alpha < 1$, define

$$B_\alpha = \left\{ u \in H_0^\alpha\backslash\{0\};\ \|u\|_\alpha^2 \leqslant \int_0^1 f(r,u)ur^\alpha \,\mathrm{d}r \right\}$$

$$\partial B_\alpha = \left\{ u \in B_\alpha;\ u \geqslant 0;\ \|u\|_\alpha^2 = \int_0^1 f(r,u)ur^\alpha \,\mathrm{d}r \right\}$$

$$B_1 = \left\{ u \in H_0^1 \cap L^\infty\backslash\{0\};\ \|u\|_1^2 \leqslant \int_0^1 f(r,u)ur \,\mathrm{d}r \right\}$$

$$B_1^* = \left\{ u \in H_0^1\backslash\{0\};\ u \text{ is non-increasing, } \|u\|_1^2 \leqslant \int_0^1 f(r,u)ur \,\mathrm{d}r \right\}$$

$$\partial(B_1 \cup B_1^*) = \left\{ u \in B_1 \cup B_1^*;\ u \geqslant 0;\ \|u\|_1^2 = \int_0^1 f(r,u)ur \,\mathrm{d}r \right\}$$

$$B_{01} = \{ u \in B_1;\ u \text{ is constant in a nhd of zero} \}.$$

$$\partial B_{01} = \left\{ u \in B_{01};\ u \geqslant 0,\ \|u\|_1^2 = \int_0^1 f(r,u)ur \,\mathrm{d}r \right\}$$

For $0 \leqslant \alpha \leqslant 1$, $f \in A'$, u in H_0^α, define

$$I_\alpha(u) = \frac{1}{2}\int_0^1 f(r,u)ur^\alpha \,dr - \int_0^1 F(r,u)r^\alpha \,dr \tag{2.13}$$

since $f \in A'$; $f(r,t)t - 2F(r,t) \geqslant 0$ for all $(r,t) \in [0,1] \times \mathbb{R}$, hence $I_\alpha(u) \geqslant 0$. Define a_α by

$$\frac{a_\alpha^2}{2} = \inf_{\Sigma_\alpha} I_\alpha. \tag{2.14}$$

Theorem 2.2. *Let f be in A'. Then there exists an $\alpha_0 < 1$ such that for $\alpha_0 \leqslant \alpha < 1$, Σ_α is non-empty and $\{a_\alpha\}$ is bounded and satisfying*

$$\frac{a_\alpha^2}{2} = \inf_{B_\alpha} I_\alpha(u) = \inf_{\partial B_\alpha} I_\alpha(u). \tag{2.15}$$

Case 1. If f is super critical then $\lim_{\alpha \to 1} a_\alpha = 0$.

Case 2. If f is critical and suppose there exists a $t_2 > 0$ such that

$$\begin{aligned} &t_2 h\left(0, \left(\frac{2}{b(0)}\right)^{1/2} t_2\right) > 2\left(\frac{2}{b(0)}\right)^{1/2} \\ &\exp(-t_2) < \delta_1 \quad [\text{see } (2.4)] \end{aligned} \tag{2.16}$$

then $\lim_{\alpha \to 1} a_\alpha = a$ exists and is non-zero. Moreover there exists u satisfying (1.2) such that

$$I_1(u) = \frac{a^2}{2} = \inf_{B_1 \cup B_1^*} I_1 = \inf_{B_{01}} I_1 = \inf_{\partial B_{01}} I_1 \tag{2.17}$$

Remark 2.1. Suppose there exists a sequence $t_n \to \infty$ such that $h(0, t_n)t_n \to \infty$, then (2.16) is satisfied.

Examples

1. *Carleson–Chang.* Let $f_\lambda(t) = \lambda t \exp(\lambda t^2)$ for $0 < \lambda < \lambda_1$. Then f_λ is in A' and satisfies (2.16). Hence (1.1) has a solution.

2. *Atkinson–Peletier.* $f(t) = t^m \exp(bt^2)$, $m > 1$, $b > 0$. Then f is in A' satisfying (2.16). Hence (1.1) has a solution.

3. $f(t) = \lambda t^m \exp(bt^2 + \sin t^2)$, $\quad b \geqslant 1$

   $m = 1, \quad 0 < \lambda < \lambda_1,$

   $m > 1, \quad \lambda > 0.$

Then f is in A' and satisfying (2.16). Hence (1.1) has a solution. Here $\log f$ is not convex for large t.

4. Let $b(r)$ be a C^1-function on $[0,1]$ such that $0 \leqslant b(r) \leqslant 1$, $b(r) \equiv 1$ in a neighbourhood of zero. Let $f(r,t) = t^m \exp(b(r)t^2 + (1 - b(r))\exp(t))$. Then f is in A' satisfying (2.16). Hence (1.1) has a solution.

3. Proofs of theorems (2.1) and (2.2)

Lemma 3.1. *For $0 \leqslant \alpha < 1$, we have*

(i) *H_0^α is compactly embedded in $C[0,1]$.*
(ii) *$\lambda_\alpha < \lambda_1$ and $\lambda_\alpha \to \lambda_1$ as $\alpha \to 1$*
(iii) *u in H_0^1, $r_1 < r_2$,*

$$|u(r_1) - u(r_2)|^2 \leqslant \|u\|_1^2 \log \frac{r_2}{r_1}.$$

Proof. Let $r_1 \leqslant r_2$ and u is in H_0^α. Then by integration by parts

$$\begin{aligned}|u(r_2) - u(r_1)|^2 &= \left(\int_{r_1}^{r_2} u'(r)\,\mathrm{d}r\right)^2 \\ &\leqslant \|u\|_\alpha^2 \int_{r_1}^{r_2} r^{-\alpha}\,\mathrm{d}r \\ &= \|u\|_\alpha^2 \frac{r_2^{1-\alpha} - r_1^{1-\alpha}}{1-\alpha}.\end{aligned} \tag{3.1}$$

Hence (i) follows from (3.1) and Arzela–Ascoli's theorem. Let u is in H_0^1, then

$$\begin{aligned}|u(r_2) - u(r_1)|^2 &= \left(\int_{r_1}^{r_2} u'(r)\,\mathrm{d}r\right)^2 \\ &\leqslant \|u\|_1^2 \left(\int_{r_1}^{r_2} r^{-1}\,\mathrm{d}r\right) \\ &= \|u\|_1^2 \log \frac{r_2}{r_1}.\end{aligned} \tag{3.2}$$

This proves (iii).

We have

$$\begin{aligned}-(r\phi_\alpha')' &= \lambda_\alpha \phi_\alpha r - (1-\alpha)\phi_\alpha' \\ -(r\phi_1')' &= \lambda_1 \phi_1 r.\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 \int_0^1 \phi_1 \phi_\alpha r\,\mathrm{d}r &= -\int_0^1 (r\phi_1')'\phi_\alpha\,\mathrm{d}r \\ &= -\int_0^1 (r\phi_1')'\phi_\alpha\,\mathrm{d}r \\ &= \lambda_\alpha \int_0^1 \phi_\alpha \phi_1 r\,\mathrm{d}r - (1-\alpha)\int_0^1 \phi_\alpha' \phi_1\,\mathrm{d}r.\end{aligned}$$

i.e.

$$(\lambda_1 - \lambda_\alpha)\int_0^1 \phi_1 \phi_\alpha r\,\mathrm{d}r = -(1-\alpha)\int_0^1 \phi_\alpha' \phi_1\,\mathrm{d}r.$$

Since $\phi'_\alpha \leqslant 0$ and hence $\lambda_\alpha \leqslant \lambda_1$ and $\lambda_\alpha \to \lambda_1$ as $\alpha \to 1$. This proves (ii).

Lemma 3.2. Let f be in A, then there exists an $\alpha_0 < 1$ such that for $\alpha_0 \leqslant \alpha < 1$,

i) *$\bar{I}_\alpha$ satisfies the Palais–Smale condition.*
ii) *Let $m > 0$ be such that*

$$\int_0^{1/2} F\left(r, \frac{m}{2}\right) r \,\mathrm{d}r \geqslant 2m^2 \tag{3.3}$$

[*Such a m exists because of the condition* (iv) *of definition* (2.1)].
Then there exists a u_α in $C^2[0,1]$ satisfying

$$\begin{aligned} L_\alpha u_\alpha &= f(r, u_\alpha) \quad \text{in } [0,1) \\ u_\alpha &> 0 \\ u'_\alpha(0) &= u_\alpha(1) = 0. \end{aligned} \tag{3.4}$$

and

$$\bar{I}_\alpha(u_\alpha) \leqslant 2m^2.$$

Proof. Proof of this lemma is standard (see [7]). For the sake of completeness we will prove it.

Step 1. Let u_n in H_0^α be a sequence such that

$$\begin{aligned} &|\bar{I}_\alpha(u_n)| \leqslant M \\ &\bar{I}'_\alpha(u_n) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\beta \bar{I}_\alpha(u_n) - \langle \bar{I}'_\alpha(u_n), u_n \rangle \\ &\quad = \left(\frac{\beta}{2} - 1\right) \int_0^1 u'_n(r)^2 r^\alpha \,\mathrm{d}r - \int_0^1 [\beta F(r, u_n) - f(r, u_n) u_n] r^\alpha \,\mathrm{d}r \\ &\quad \geqslant \left(\frac{\beta}{2} - 1\right) \int_0^1 u'_n(r)^2 r^\alpha \,\mathrm{d}r - \int_{|u_n| \leqslant t_1} [\beta F(r, u_n) - f(r, u_n) u_n] r^\alpha \,\mathrm{d}r \\ &\quad \geqslant \left(\frac{\beta}{2} - 1\right) \int_0^1 u'_n(r)^2 r^\alpha \,\mathrm{d}r + C, \end{aligned} \tag{3.6}$$

where C is a constant depending only on F. Since $\beta > 2$, (3.5) and (3.6) imply $\{\|u_n\|_\alpha\}$ is bounded. Let u_n converge to u weakly in H_0^α and strongly in $C[0,1]$.

$$\begin{aligned} \langle \bar{I}'(u_n), u_n - u \rangle = &\int_0^1 u'_n(r)^2 r^\alpha \,\mathrm{d}r - \int_0^1 u'_n(r) u'(r) r^\alpha \,\mathrm{d}r \\ &- \int_0^1 f(r, u_n)(u_n - u) r^\alpha \,\mathrm{d}r \end{aligned} \tag{3.7}$$

(3.5) and (3.7) imply

$$\int_0^1 u'_n(r)^2 r^\alpha \,\mathrm{d}r \to \int_0^1 u'(r)^2 r^\alpha \,\mathrm{d}r.$$

Hence u_n converges strongly to u and this proves (i).

Step 2. From (ii) of Lemma (3.1) and (iii) of Definition (2.1) there exists an $\alpha_0 < 1$ and a $\lambda > 0$ such that

$$F(r,t) \leqslant \frac{\lambda t^2}{2} < \frac{\lambda_\alpha t^2}{2} \quad \text{for all } r \in [0,1], \quad 0 < |t| < t_0. \tag{3.8}$$

Let u in H_0^α be such that

$$\|u\|_\alpha^2 \leqslant \frac{(1-\alpha)}{2} t_0^2. \tag{3.9}$$

From (3.1) and (3.9) we have

$$|u(r)|^2 \leqslant t_0^2. \tag{3.10}$$

Hence (3.8) and (3.10) give

$$F(r,u(r)) \leqslant \frac{\lambda u(r)^2}{2} \tag{3.11}$$

$$\begin{aligned} \bar{I}_\alpha(u) &= \tfrac{1}{2}\|u\|_\alpha^2 - \int_0^1 F(r,u) r^\alpha \, dr \\ &\geqslant \tfrac{1}{2}\|u\|_\alpha^2 - \frac{\lambda}{2}\int_0^1 u(r)^2 r^\alpha \, dr \\ &\geqslant \tfrac{1}{2}\left[\|u\|_\alpha^2 - \frac{\lambda}{\lambda_\alpha}\|u\|_\alpha^2 \right] \\ &= \tfrac{1}{2}\left(1 - \frac{\lambda}{\lambda_\alpha}\right)\|u\|_\alpha^2. \end{aligned} \tag{3.12}$$

Hence zero is a local minima.

Step 3. Define u_0 in H_0^0 by

$$u_0(r) = \begin{cases} \dfrac{m}{2} & 0 \leqslant r < \tfrac{1}{2} \\ m(1-r) & \tfrac{1}{2} \leqslant r \leqslant 1 \end{cases} \tag{3.13}$$

Then

$$\begin{aligned} \bar{I}_\alpha(u_0) &= \frac{1}{2}\int_{1/2}^1 m^2 r^\alpha \, dr - \int_0^1 F(r,u_0) r^\alpha \, dr \\ &\leqslant \frac{m^2}{2(1+\alpha)}\left(1 - \frac{1}{2^{1+\alpha}}\right) - \int_0^{1/2} F(r,u_0) r^\alpha \, dr \\ &\leqslant \frac{m^2}{2(1+\alpha)}\left(1 - \frac{1}{2^{1+\alpha}}\right) - \int_0^{1/2} F\left(r,\frac{m}{2}\right) r^\alpha \, dr \\ &\leqslant \frac{m^2}{2(1+\alpha)}\left(1 - \frac{1}{2^{1+\alpha}}\right) - 2m^2 < 0 \end{aligned} \tag{3.14}$$

and for $0 \leqslant t \leqslant 1$,

$$\bar{I}_\alpha(tu_0) \leqslant \frac{t^2}{2} \| u_0 \|_\alpha^2$$

$$\leqslant \frac{m^2}{2(1+\alpha)} \left(1 - \frac{1}{2^{1+\alpha}}\right) \leqslant 2m^2 \tag{3.15}$$

Hence $\bar{I}_\alpha$ satisfies all the hypotheses of Mountain pass theorem and hence there exists a critical point u_α of $\bar{I}_\alpha$ such that

$$\bar{I}_\alpha(u_\alpha) \leqslant \sup_{t \in [0,1]} \bar{I}_\alpha(tu_0).$$

Now from (3.15) it follows that

$$\bar{I}_\alpha(u_\alpha) \leqslant 2m^2$$

and u_α satisfies (3.4).

Lemma 3.3. Let f be in A', then there exists $\alpha_0 < 1$ such that for all $\alpha_0 \leqslant \alpha < 1$, $\sum_\alpha$ is non-empty and an $u_\alpha \in \sum_\alpha$ satisfying

$$\frac{a_\alpha^2}{2} = I_\alpha(u_\alpha) = \inf_{u \in \partial B_\alpha} I_\alpha(u) = \inf_{u \in B_\alpha} I_\alpha(u) \tag{3.16}$$

and for all w in H_0^α, $\| w \|_\alpha = 1$,

$$\int_0^1 f(r, a_\alpha w) w r^\alpha \, dr \leqslant a_\alpha. \tag{3.17}$$

Proof. Let u be in B_α. Define $\gamma \leqslant 1$ such that

$$\| u \|_\alpha^2 = \frac{1}{\gamma} \int_0^1 f(r, \gamma u) u r^\alpha \, dr. \tag{3.18}$$

Such a γ exists because $f(r,t)/t$ is an increasing function and u is in B_α and $|f(r,t)| < \lambda_\alpha |t|$ for $|t| < t_0$; $\alpha_0 \leqslant \alpha < 1$.

Define $v = \gamma u$, then

$$\| v \|_\alpha^2 = \gamma^2 \| u \|_\alpha^2 = \int_0^1 f(r, \gamma u)(\gamma u) r^\alpha \, dr$$

$$= \int_0^1 f(r, v) v r^\alpha \, dr. \tag{3.19}$$

Hence v is in ∂B_α and since $\gamma \leqslant 1$, and $f \in A'$, we have

$$I_\alpha(v) = I_\alpha(\gamma u) \leqslant I_\alpha(u).$$

this together with $\partial B_\alpha \subset B_\alpha$ imply that

$$d_\alpha = \inf_{\partial B_\alpha} I_\alpha = \inf_{B_\alpha} I_\alpha. \tag{3.20}$$

Let u_n in ∂B_α be a sequence such that $u_n \geqslant 0$ and $I_\alpha(u_n) \to d_\alpha$. Such a sequence exists because for u in ∂B_α implies $|u|$ is in ∂B_α and $I_\alpha(u) = I_\alpha(|u|)$.

We claim that $\{\|u_n\|_\alpha\}$ is bounded. Let N be such that for all $n \geqslant N$,

$$d_\alpha \leqslant I_\alpha(u_n) \leqslant d_\alpha + 1 \tag{3.21}$$

$$\begin{aligned} d_\alpha + 1 \geqslant I_\alpha(u_n) &= \frac{1}{2}\int_0^1 [f(r, u_n)u_n - 2F(r, u_n)]r^\alpha \, \mathrm{d}r \\ &= \frac{1}{2}\int_0^1 [f(r, u_n)u_n - \beta F(r, u_n)]r^\alpha \, \mathrm{d}r \\ &\quad + \left(\frac{\beta}{2} - 1\right)\int_0^1 F(r, u_n)r^\alpha \, \mathrm{d}r. \end{aligned} \tag{3.22}$$

From (iv) of Definition (2.1), there exists a constant C depending only on f such that for all v in H_0^α,

$$\int_0^1 [f(r, v)v - \beta F(r, v)]r^\alpha \, \mathrm{d}r \geqslant C. \tag{3.23}$$

From (3.22) and (3.23) there exists a constant C_1 independent of n such that

$$\int_0^1 F(r, u_n)r^\alpha \, \mathrm{d}r \leqslant C_1. \tag{3.24}$$

From (3.21) and (3.24) we have

$$\begin{aligned} \|u\|_\alpha^2 &= 2I_\alpha(u_n) + 2\int_0^1 F(r, u_n)r^\alpha \, \mathrm{d}r \\ &\leqslant 2(d_\alpha + 1) + 2C_1 \end{aligned}$$

and this proves the claim.

Let $u_\alpha =$ weak limit of u_n and α_0 be as in Lemma (3.2). We claim that for $\alpha_0 \leqslant \alpha < 1$, $u_\alpha \in \sum_\alpha$ satisfying (3.16).

First we will show that u_α is non-zero. Suppose $u_\alpha \equiv 0$, then from Lemma (3.1). u_n converges to 0 in $C[0, 1]$. Let N be an integer such that

$$u_n(r) < t_0 \quad \text{for all } n \geqslant N, r \in [0, 1]. \tag{3.25}$$

Then from (iii) of Definition (2.1) and the choice of α_0,

$$f(r, u_n(r)) < \lambda_\alpha u_n(r). \tag{3.26}$$

Since $u_n \in \partial B_\alpha$, we have from (3.26)

$$\begin{aligned} \|u_n\|_\alpha^2 &= \int_0^1 f(r, u_n)u_n r^\alpha \, \mathrm{d}r \\ &< \lambda_\alpha \int_0^1 u_n(r)^2 r^\alpha \, \mathrm{d}r \leqslant \|u_n\|_\alpha^2 \end{aligned}$$

which is a contradiction and hence $u_\alpha \not\equiv 0$ and

$$I_\alpha(u_\alpha) = \lim_{n\to\infty} I_\alpha(u_n) = d_\alpha$$

$$\| u_\alpha \|_\alpha^2 \leqslant \varliminf_{n\to\infty} \| u_n \|_\alpha^2 = \int_0^1 f(r, u_\alpha) u_\alpha r^\alpha \, \mathrm{d}r, \tag{3.27}$$

u_α is in ∂B_α. If not, then by (3.27) we can choose a $\gamma < 1$ such that

$$\| u_\alpha \|^2 = \frac{1}{\gamma} \int_0^1 f(r, \gamma u_\alpha) u_\alpha r^\alpha \, \mathrm{d}r.$$

Then γu_α is in ∂B_α and

$$d_\alpha \leqslant I(\gamma u_\alpha) < I(u_\alpha) = d_\alpha.$$

This proves that u_α is in ∂B_α. Since u_α is a minimizer and hence there exists a real number ρ such that for all ϕ in H_0^α,

$$\int_0^1 u_\alpha'(r)\phi'(r) r^\alpha \, \mathrm{d}r - \int_0^1 f(r, u_\alpha)\phi r^\alpha \, \mathrm{d}r$$
$$= \rho \left\{ 2 \int_0^1 u_\alpha'(r)\phi'(r) r^\alpha \, \mathrm{d}r - \int_0^1 f(r, u_\alpha)\phi r^\alpha \, \mathrm{d}r - \int_0^1 \frac{\partial f}{\partial t}(r, u_\alpha) u_\alpha \phi r^\alpha \, \mathrm{d}r \right\}. \tag{3.28}$$

Putting $\phi = u_\alpha$ in (3.28) and using the fact that $u_\alpha \in \partial B_\alpha$, we have

$$\rho \left\{ 2 \int_0^1 u_\alpha'(r)^2 r^\alpha \, \mathrm{d}r - \int_0^1 f(r, u_\alpha) u_\alpha r^\alpha \, \mathrm{d}r - \int_0^1 \frac{\partial f}{\partial t}(r, u_\alpha) u_\alpha(r)^2 r^\alpha \, \mathrm{d}r \right\} = 0.$$

Since u_α is in ∂B_α, we have

$$\rho \int_0^1 \left[\frac{f(r, u_\alpha)}{u_\alpha} - \frac{\partial f}{\partial t}(r, u_\alpha) \right] u_\alpha(r)^2 r^\alpha \, \mathrm{d}r = 0.$$

Since f is in A', and u is not zero, it implies that $\rho = 0$. Hence from (3.28) and by regularity of elliptic operator, it follows that u_α is in $\sum_\alpha$ and $I_\alpha(u_\alpha) = d_\alpha$. Since $\sum_\alpha \subset \partial B_\alpha$, we have $a_\alpha^2/2 = \inf_{\sum_\alpha} I_\alpha = I_\alpha(u_\alpha) = d_\alpha$ and this proves (3.16). Let $\| w \|_\alpha = 1$. Choose $\gamma > 0$ such that

$$1 = \frac{1}{\gamma} \int_0^1 f(r, \gamma w) w r^\alpha \, \mathrm{d}r. \tag{3.29}$$

Then γw is in ∂B_α. Hence

$$\frac{a_\alpha^2}{2} \leqslant I_\alpha(\gamma w) \leqslant \frac{\gamma^2}{2} \| w \|_\alpha^2 = \frac{\gamma^2}{2}$$

implies $a_\alpha \leqslant \gamma$. Since f is in A', we have

$$\frac{1}{a_\alpha}\int_0^1 f(r, a_\alpha w)wr^\alpha\,dr \leqslant \frac{1}{\gamma}\int_0^1 f(r,\gamma w)wr^\alpha\,dr = 1$$

i.e.

$$\int_0^1 f(r, a_\alpha w)wr^\alpha\,dr \leqslant a_\alpha$$

proving (3.17).

Lemma 3.4. Let f be in A' and α_0 is as in Lemma (3.3). Then $\{a_\alpha\}$ is bounded on $[\alpha_0, 1)$. Let $a = \overline{\lim}_{\alpha\to 1} a_\alpha$. Then for all $w \in H_0^1$ with $\|w\|_1 = 1$, we have

$$\int_0^1 f(r, aw)wr\,dr \leqslant a. \tag{3.30}$$

Proof. From Lemma (3.2) and (3.3) we have $l_\alpha = a_\alpha^2/2$ and $l_\alpha \leqslant 2m^2$. Hence $\{a_\alpha\}$ is bounded on $[\alpha_0, 1)$. Let α_n be a sequence such that $a_{\alpha_n} \to a$ as $\alpha_n \to 1$ and w be in E with $\|w\|_1 = 1$. Let $v_n = w/\|w\|_{\alpha_n}$. Then from (3.17) we have

$$\int_0^1 f(r, a_{\alpha_n} v_n)v_n r^\alpha\,dr \leqslant a_{\alpha_n}.$$

Letting $\alpha_n \to 1$, $v_n \to w$, $a_{\alpha_n} \to a$, we get

$$\int_0^1 f(r, aw)wr\,dr \leqslant a. \tag{3.31}$$

Since f is odd, and hence by Fatou's (3.31) holds for all w in H_0^1.

Lemma 3.5. Let f be in A, $0 \leqslant \alpha < 1$, $0 \leqslant \varepsilon \leqslant 1$, and u in $\sum_\alpha$. Then we have

$$u(r) = \frac{1 - r^{1-\alpha}}{1-\alpha}\int_0^r f(t, u(t))t^\alpha\,dt + \int_r^1 t^\alpha\left(\frac{1 - t^{1-\alpha}}{1-\alpha}\right) f(t, u(t))\,dt \tag{3.32}$$

$$\begin{aligned}\tfrac{1}{2}\varepsilon^{1+\alpha}u'(\varepsilon)^2 = (1+\alpha)\int_0^\varepsilon F(r,u)r^\alpha\,dr + \int_0^\varepsilon \frac{\partial F}{\partial r}(r,u)r^{1+\alpha}\,dr \\ + \frac{1-\alpha}{2}\int_0^\varepsilon u'(r)^2 r^\alpha\,dr - \varepsilon^{1+\alpha}F(\varepsilon, u(\varepsilon)).\end{aligned} \tag{3.33}$$

Proof. If $v(r)$ is the right hand side of (3.32), then by differentiating twice, v satisfies

$$\begin{aligned} L_\alpha v &= f(r,u) \\ v'(0) &= v(1) = 0. \end{aligned} \tag{3.34}$$

Hence by uniqueness, $v = u$. This proves (3.32). u is in $\sum_\alpha$, hence

$$(r^\alpha u')' = -f(r, u(r))r^\alpha. \tag{3.35}$$

multiply (3.35) by $ru'(r)$ and integrate from 0 to ε we get

$$\int_0^\varepsilon (r^\alpha u'(r))' u'(r) r \,\mathrm{d}r = -\int_0^\varepsilon f(r,u) u' r^{1+\alpha} \,\mathrm{d}r. \tag{3.36}$$

Since $(\mathrm{d}F/\mathrm{d}r)(r,u(r)) = (\partial F/\partial r)(r,u(r)) + f(r,u(r))u'(r)$, we have

$$\begin{aligned} \tfrac{1}{2}\varepsilon^{1+\alpha}u'(\varepsilon)^2 - \frac{(1-\alpha)}{2}\int_0^\varepsilon u'(r)^2 r^\alpha \,\mathrm{d}r &= -\int_0^\varepsilon \frac{\mathrm{d}F}{\mathrm{d}r} r^{1+\alpha}\,\mathrm{d}r + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+\alpha}\,\mathrm{d}r \\ &= -F(\varepsilon,u(\varepsilon))\varepsilon^{1+\alpha} + (1+\alpha)\int_0^\varepsilon F(r,u) r^\alpha \,\mathrm{d}r \\ &\quad + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+\alpha}\,\mathrm{d}r. \end{aligned}$$

Hence

$$\begin{aligned} \tfrac{1}{2}\varepsilon^{1+\alpha}u'(\varepsilon)^2 = (1+\alpha)\int_0^\varepsilon F(r,u)r^\alpha\,\mathrm{d}r + \int_0^\varepsilon \frac{\partial F}{\partial r} r^{1+\alpha}\,\mathrm{d}r + \frac{1-\alpha}{2}\int_0^\varepsilon u'(r)^2 r^\alpha\,\mathrm{d}r \\ - F(\varepsilon,u(\varepsilon))\varepsilon^{1+\alpha}. \end{aligned}$$

This proves (3.33).

Lemma 3.6. Let f be in A, $\alpha_n \to 1$, u_n is in $\sum_{\alpha_n}$ and a constant M independent of n such that

$$\begin{aligned} &\text{(i)}\ \ \|u_n\|_{\alpha_n} \leqslant M \\ &\text{(ii)}\ \ \lim_{n\to\infty} u_n'(1) = \eta \neq 0. \end{aligned} \tag{3.37}$$

Then there exists a subsequence (still denoted by α_n) such that the weak limit u of u_n in H_0^1 is a weak solution of (1.2). *Furthermore*

$$\lim_{n\to\infty}\int_0^1 F(r,u_n) r^{\alpha_n}\,\mathrm{d}r = \int_0^1 F(r,u) r\,\mathrm{d}r. \tag{3.38}$$

Proof. $\|u_n\|_1 \leqslant \|u_n\|_{\alpha_n} \leqslant M$, hence by going to a subsequence the weak limit u of u_n in H_0^1 exists. From (iii) of Lemma (3.1), u_n converges to u uniformly on compact subsets of $(0,1]$. We claim u is not identically zero. For, if $u \equiv 0$, then, since u_n in $\sum_{\alpha_n}$, we have for $0 < r \leqslant 1$,

$$r^{\alpha_n}u_n'(r) = u_n'(1) + \int_r^1 f(r,u_n) r^{\alpha_n}\,\mathrm{d}r. \tag{3.39}$$

From (ii) of (3.37) and (3.39) and using $u_n \to 0$ on $[r,1]$ uniformly

$$r \lim_{n\to\infty} u_n'(r) = \eta. \tag{3.40}$$

Hence by Fatou's lemma, and (3.40)

$$\infty = \eta^2 \int_0^1 \frac{r\,\mathrm{d}r}{r^2} < \int_0^1 \varliminf_{n\to\infty} u_n'(r)^2 r^{\alpha_n}\,\mathrm{d}r \leqslant \varliminf \|u_n\|_{\alpha_n}^2 \leqslant M$$

which is a contradiction. Hence $u \not\equiv 0$ and u satisfies

$$\begin{aligned} -(ru')' &= f(r,u)r \quad \text{in } (0,1] \\ u(1) &= 0. \end{aligned} \tag{3.41}$$

Now by Fatous, we have

$$\int_0^1 f(r,u)ur\,\mathrm{d}r \leqslant \varliminf \int_0^1 f(r,u_n)u_n r^\alpha\,\mathrm{d}r \leqslant M. \tag{3.42}$$

Hence

$$\int_0^1 f(r,u)r\,\mathrm{d}r \leqslant \int_{u\leqslant 1} f(r,u)r\,\mathrm{d}r + \int_{u>1} f(r,u)ur\,\mathrm{d}r < \infty. \tag{3.43}$$

For any $0<r\leqslant 1$, integrating (3.41) from r to 1, we get

$$ru'(r) = u'(1) + \int_r^1 f(t,u)t\,\mathrm{d}t. \tag{3.44}$$

(3.44) gives $ru'(r)$ is monotone and hence limit $r\to 0$ exists. We claim that

$$\lim_{r\to 0} ru'(r) = 0. \tag{3.45}$$

For, if $\lim_{r\to 0} ru'(r) = C < 0$, then there exists $\varepsilon > 0$ such that $-u'(r) \geqslant C/r$ for $0<r\leqslant\varepsilon$. Hence

$$\infty = C^2\int_0^\varepsilon \frac{r\,\mathrm{d}r}{r^2} \leqslant \int_0^\varepsilon ru'(r)^2\,\mathrm{d}r < \infty.$$

Hence (3.45) is true. Using (3.44) and (3.45) we get

$$u'(1) = -\int_0^1 f(t,u)t\,\mathrm{d}t. \tag{3.46}$$

Let ϕ be in $C^2[0,1]$ with $\phi(1)=0$. Multiply ϕ' to (3.44) and integrate from 0 to 1, and using (3.46) we have

$$\begin{aligned} \int_0^1 u'(r)\phi'(r)r\,\mathrm{d}r &= u'(1)(\phi(1)-\phi(0)) + \int_0^1 \phi'(r)\int_r^1 f(t,u)t\,\mathrm{d}t\,\mathrm{d}r \\ &= u'(1)(\phi(1)-\phi(0)) + \int_0^1 f(t,u)\phi(t)t\,\mathrm{d}t \\ &\quad - \phi(0)\int_0^1 f(t,u)t\,\mathrm{d}t \\ &= \int_0^1 f(t,u)\phi(t)t\,\mathrm{d}t \end{aligned}$$

and hence u is a weak solution of (1.2).

From (3.33) and (3.37) we have

$$\lim_{n\to\infty}\left\{(1+\alpha_m)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}r^{1+\alpha_n}\,\mathrm{d}r\right\}=\tfrac{1}{2}\eta^2 \tag{3.47}$$

Now multiply $ru'(r)$ to (3.41) and integrate from r to 1, we have

$$\begin{aligned}-\tfrac{1}{2}r^2u'(r)^2+\tfrac{1}{2}u'(1)^2&=-\int_r^1\frac{\mathrm{d}F}{\mathrm{d}t}t^2\,\mathrm{d}t+\int_{r_1}^1\frac{\partial F}{\partial t}t^2\,\mathrm{d}t\\&=F(r,u(r))r^2+2\int_r^1 F(t,u)t+\int_r^1\frac{\partial F}{\partial t}t^2\,\mathrm{d}t.\end{aligned} \tag{3.48}$$

Since $ru'(r)\to 0$, $\int_0^1 F(t,u)t\,\mathrm{d}t<\infty$, $\partial F/\partial r>0$ in $[0,\delta_0]$ and $\int_{\delta_0}^1(\partial F/\partial t)t^2\,\mathrm{d}t<\infty$, we conclude that $\lim_{r\to 0}F(r,u(r))r^2$ exists and claim that

$$\lim_{r\to 0}F(r,u(r))r^2=0. \tag{3.49}$$

If not, there exists a constant $C>0$ and $\varepsilon>0$ such that

$$F(r,u(r))r^2\geqslant C\quad\text{for all } 0<r<\varepsilon.$$

Hence

$$\infty=\int_0^\varepsilon\frac{C}{r}\,\mathrm{d}r\leqslant\int_0^\varepsilon F(r,u(r))r\,\mathrm{d}r<\infty$$

which is a contradiction.

Now using (3.49), (3.48) becomes

$$\tfrac{1}{2}u'(1)^2=2\int_0^1 F(r,u)r\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}(r,u)r^2\,\mathrm{d}r. \tag{3.50}$$

Since $u'(1)=\lim_{n\to\infty}u_n'(1)$, and hence from (3.47) and (3.50) we have

$$\begin{aligned}&2\int_0^1 F(r,u)r\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}(r,u)r^2\,\mathrm{d}r\\&\quad=\lim_{n\to\infty}\left\{(1+\alpha_n)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r+\int_0^1\frac{\partial F}{\partial r}(r,u_n)r^{1+\alpha_n}\,\mathrm{d}r\right\}.\end{aligned} \tag{3.51}$$

By Fatou's and using (ii) of Definition (2.1) we have

$$\begin{aligned}&2\int_0^1 F(r,u)r\,\mathrm{d}r\leqslant\underline{\lim}\,(1+\alpha_n)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r\\&\int_0^1\frac{\partial F}{\partial r}(r,u)r^2\,\mathrm{d}r\leqslant\underline{\lim}\int_0^1\frac{\partial F}{\partial r}(r,u_n)r^{\alpha_n+1}\,\mathrm{d}r.\end{aligned} \tag{3.52}$$

By going to a subsequence, we conclude from (3.51) and (3.52) that

$$\lim_{n\to\infty}(1+\alpha_n)\int_0^1 F(r,u_n)r^{\alpha_n}\,\mathrm{d}r=2\int_0^1 F(r,u)r\,\mathrm{d}r$$

and

$$\lim_{n\to\infty}\int_0^1 \frac{\partial F}{\partial r}(r,u_n)r^{\alpha_n+1}\,dr = \int_0^1 \frac{\partial F}{\partial r}(r,u)r^2\,dr.$$

Lemma 3.7. Let f in A' be critical. Then

$$\frac{2}{b(0)} = \sup\left\{c^2;\ \sup_{\|w\|_1\leqslant 1}\int_0^1 f(r,cw)wr\,dr < \infty\right\} \tag{3.53}$$

Proof. $f = h(r,t)\exp[b(r)t^2]$ for (r,t) in Q_{δ_1}.
Let

$$C_0^2 = \sup\left\{c^2;\ \sup_{\|w\|_1\leqslant 1}\int_0^1 f(r,cw)wr\,dr < \infty\right\}$$

Step 1. $C_0^2 \not> 2/b(0)$.
If not, then choose $\varepsilon > 0$, $c > 0$ and a $\delta < (\delta_1, \delta_0)$ such that

$$\frac{2}{b(0)} < c^2 < (c+\varepsilon)^2 < C_0^2. \tag{3.54}$$

For $r_0 \in [0, \delta_1]$, define

$$\begin{aligned} W_{r_0}(r) &= \frac{\log\frac{1}{r}}{\left(\log\frac{1}{r_0}\right)^{1/2}} \quad \text{for } r_0 \leqslant r \leqslant 1 \\ W_{r_0}(r) &= \left(\log\frac{1}{r_0}\right)^{1/2} \quad \text{for } 0 \leqslant r \leqslant r_0. \end{aligned} \tag{3.55}$$

Then $\|w_{r_0}\|_1 = 1$. Since $(\partial f/\partial r)(r,t) \geqslant 0$ in Q_{δ_0}, we have

$$h(0,t)\exp[b(0)t^2] \leqslant h(r,t)\exp[b(r)t^2] \quad \text{in } Q_{\delta_0}.$$

$(c+\varepsilon)^2 < C_0^2$ implies that there exists an absolute constant M depending only) and f such that

$$\begin{aligned} M &\geqslant \int_0^1 f(r,(c+\varepsilon)w_{r_0})w_{r_0}r\,dr \geqslant \int_0^\delta f(r,(c+\varepsilon)w_{r_0})w_{r_0}r\,dr \\ &\geqslant \int_0^{r_0} f\left(0,(c+\varepsilon)\left(\log\frac{1}{r_0}\right)^{1/2}\right)\left(\log\frac{1}{r_0}\right)^{1/2} r\,dr \\ &= \frac{1}{2}\left(\log\frac{1}{r_0}\right)^{1/2} h\left(0,(c+\varepsilon)\left(\log\frac{1}{r_0}\right)^{1/2}\right)\exp\left[b(0)(c+\varepsilon)^2\log\frac{1}{r_0}\right]r_0^2 \\ &\geqslant \frac{\frac{1}{2}\left(\log\frac{1}{r_0}\right)^{1/2} h\left(0,(c+\varepsilon)\left(\log\frac{1}{r_0}\right)^{1/2}\right)\exp\left[\varepsilon^2\left(\log\frac{1}{r_0}\right)\right]}{r_0^{(c^2b(0)-2)}} \to \infty \end{aligned}$$

as $r_0 \to 0$.

Hence $C_0^2 \leqslant 2/b(0)$.

Step 2. $C_0^2 = 2/b(0)$.
Suppose not, then choose $\varepsilon > 0$, $\delta > 0$ such that $\delta \leqslant \min(\delta_1, \delta_0)$ and for all r in $[0, \delta]$,

$$C_0^2 < (C_0 + \varepsilon)^2 < \frac{2-\varepsilon}{b(r)}.$$

Let $\| w \|_1 \leqslant 1$, then

$$\int_0^1 f(r, (C_0 + \varepsilon)w)wr\,\mathrm{d}r = \int_0^\delta + \int_\delta^1. \tag{3.56}$$

Since $\| w \|_1 = 1$ implies from Lemma (3.1)

$$|w(r)| \leqslant \log\frac{1}{r},$$

hence there exists a constant M_1 such that

$$\sup_{\|w\|_1 \leqslant 1} \int_\delta^1 f(r, (C_0 + \varepsilon)w)wr\,\mathrm{d}r \leqslant M_1 \tag{3.57}$$

and

$$\begin{aligned}\int_0^\delta f(r, (C_0 + \varepsilon)w)wr\,\mathrm{d}r &\leqslant \int_0^\delta h(r, (C_0 + \varepsilon)w)[\exp(C_0 + \varepsilon)^2 b(r) w^2]wr\,\mathrm{d}r \\ &\leqslant \int_0^\delta h(r, (C_0 + \varepsilon)w)[\exp(2 - \varepsilon)w^2]wr\,\mathrm{d}r \\ &\leqslant M_2 \int_0^\delta [\exp(2 - \varepsilon/2)w^2]r\,\mathrm{d}r \\ &\leqslant M_2 \int_0^\delta r^{\varepsilon/2 - 1}\,\mathrm{d}r \leqslant M_3\end{aligned} \tag{3.58}$$

where

$$M_2 = \sup_{(r,t)\in Q_\delta} h(r,t)t \exp -\frac{\varepsilon}{2}t^2.$$

This implies $C_0 > (C_0 + \varepsilon)$ which is a contradiction. Hence $C_0^2 = 2/b(0)$.

Lemma 3.8. Let f in A' be critical and suppose there exists a $t_0 > 0$ satisfying

$$\exp - t_0^2 < \delta_1$$

$$h\left(0, \left(\frac{2}{b(0)}\right)^{1/2}\right)t_0 > 2\left(\frac{2}{b(0)}\right)^{1/2} \tag{3.59}$$

Let $a \geqslant 0$ such that

$$\sup_{\|w\|_1 \leqslant 1} \int_0^1 f(r, aw) wr \, \mathrm{d}r \leqslant a \tag{3.60}$$

then $a^2 < 2/b(0)$.

Proof. From Lemma (3.7), $a^2 \leqslant 2/b(0)$. Suppose $a^2 = 2/b(0)$, then take $r_0 = \exp - t_0^2$, w_{r_0} as in (3.55) and from (3.60) we have

$$\begin{aligned}
\left(\frac{2}{b(0)}\right)^{1/2} = a &\geqslant \int_0^{r_0} f(r, aw_{r_0}) w_{r_0} r \, \mathrm{d}r \\
&\geqslant \int_0^{r_0} f(0, aw_{r_0}) w_{r_0} \, \mathrm{d}r \\
&= f(0, at_0) t_0 \frac{r_0^2}{2} \\
&= t_0 h(0, at_0) \exp 2\left(\log \frac{1}{r_0}\right) \frac{r_0^2}{2} \\
&= \tfrac{1}{2} t_0 h\left(0, \left(\frac{2}{b(0)}\right)^{1/2} t_0\right) > \left(\frac{2}{b(0)}\right)^{1/2}
\end{aligned}$$

which is a contradiction. Hence the result.

Lemma 3.9. For any $\varepsilon > 0$, $0 \leqslant \alpha < 1$,

$$\sup_{0 \leqslant r \leqslant 1} r^{\varepsilon} \left(\frac{1 - r^{1-\alpha}}{1 - \alpha}\right) \leqslant \frac{1}{\varepsilon}. \tag{3.61}$$

Proof. Let $g(r) = r^{\varepsilon}(1 - r^{1-\alpha}/1 - \alpha)$. Then $g(0) = g(1) = 0$. Let $0 < r_0 < 1$ such that

$$g(r_0) = \sup_{0 \leqslant r \leqslant 1} g(r)$$

then

$$0 = g'(r_0) = \varepsilon r_0^{\varepsilon - 1} \left(\frac{1 - r_0^{1-\alpha}}{1 - \alpha}\right) - r_0^{\varepsilon - \alpha}.$$

Hence

$$\frac{1 - r_0^{1-\alpha}}{1 - \alpha} = \frac{r_0^{1-\alpha}}{\varepsilon}.$$

Therefore

$$g(r) \leqslant g(r_0) \leqslant \frac{r_0^{1-\alpha+\varepsilon}}{\varepsilon} \leqslant \frac{1}{\varepsilon}.$$

Lemma 3.10. Let f in A′ be critical, then

$$\inf_{B_1 \cup B_1^*} I_1 = \inf_{\partial(B_1 \cup B_1^*)} I_1 = \inf_{B_{01}} I_1 \tag{3.62}$$

Proof. u is in $B_1 \cup B_1^*$ implies $|u|$ also in $B_1 \cup B_1^*$ and $I_1(u) = I_1(|u|)$. Let $u \in B_1 \cup B_1^*$; choose a $\gamma < 1$ such that

$$\|u\|_1^2 = \frac{1}{\gamma} \int_0^1 f(r, \gamma u) ur \, \mathrm{d}r.$$

Then γu is in $\partial(B_1 \cup B_1^*)$ and $I_1(\gamma u) \leqslant I_1(u)$. Hence

$$\inf_{B_1 \cup B_1^*} I_1 = \inf_{\partial(B_1 \cup B_1^*)} I_1.$$

Now let $u \geqslant 0$ is in $\partial(B_1 \cup B_1^*)$. Since f is critical, we have for any $s > 1$

$$\int_0^1 f(r, su) ur \, \mathrm{d}r < \infty.$$

Let $v = su$, then

$$\begin{aligned} \|v\|_1^2 = s^2 \|u\|_1^2 &= s^2 \int_0^1 f(r, u) ur \, \mathrm{d}r \\ &= s \int_0^1 f\left(r, \frac{v}{s}\right) vr \, \mathrm{d}r < \int_0^1 f(r, v) vr \, \mathrm{d}r \end{aligned} \tag{3.63}$$

because $s > 1$ and $f(r, t)/t$ is increasing.

Choose an $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$

$$\|v\|_1^2 < \int_\varepsilon^1 f(r, v) vr \, \mathrm{d}r \leqslant \int_0^1 f(r, v) vr \, \mathrm{d}r \tag{3.64}$$

and define

$$v_\varepsilon = \begin{cases} v(\varepsilon) & \text{if } 0 \leqslant r \leqslant \varepsilon \\ v(r) & \text{if } \varepsilon \leqslant r \leqslant 1. \end{cases} \tag{3.65}$$

Then from (3.64) v_ε is in B_{01}.

Now we claim that $I_1(v_\varepsilon) \to I_1(v)$ as $\varepsilon \to 0$.

Case 1. If v is in B_1, then $\|v_\varepsilon\|_\infty \leqslant \|v\|_\infty$ and hence by dominated convergence theorem $I_1(v_\varepsilon) \to I_1(v)$.

Case 2. If v is in B_1^*, then $v_\varepsilon \uparrow v$ and hence by Monotone convergence theorem, $I_1(v_\varepsilon) \to I_1(v)$. Hence

$$\inf_{B_{01}} I_1 \leqslant I_1(v_\varepsilon) \to I_1(v) \quad \text{as } \varepsilon \to 0. \tag{3.66}$$

f is critical and is in A', we have for $1 \leqslant s \leqslant 2$

$$f(r, su)su - 2F(r, su) \leqslant 2f(r, 2u)u - 2F(r, 2u)$$

and is in L^1. Hence by dominated convergence theorem,

$$I_1(v) \to I_1(u) \quad \text{as } s \to 1. \tag{3.67}$$

Combining (3.66) and (3.67) we have

$$\inf_{B_{01}} I_1 \leqslant \inf_{\partial(B_1 \cup B_1^*)} I_1 \leqslant \inf_{B_{01}} I_1$$

and hence the result.

Proof of theorem (2.1). From Lemma (3.2), there exists $\alpha_0 < 1$ such that Σ_α is non-empty for $\alpha_0 \leqslant \alpha < 1$ and $\{l_\alpha\}$ is bounded by $2m^2$ where m is given by (3.3). Let $l = \lim_{\alpha \to 1} l_\alpha$.

Let f satisfies (2.10). Let $\eta > 0$, $\gamma > 0$, $\alpha_n \to 1$, u_n in Σ_{α_n} such that

$$\begin{aligned} &\text{(i) } l_{\alpha_n} \to l \quad \text{as } \alpha_n \to 1 \\ &\text{(ii) } l_{\alpha_n} \leqslant \bar{I}_{\alpha_n}(u_n) < \left(l_{\alpha_n} + \frac{\eta}{2} \right). \\ &\text{(iii) } (l_{\alpha_n} + \eta) b \leqslant \gamma < 1. \end{aligned} \tag{3.68}$$

We claim that

$$\lim_{\alpha_n \to 1} u_n'(1) \neq 0. \tag{3.69}$$

If not, then $u_n'(1) \to 0$. Since $u_n \in \Sigma_{\alpha_n}$, we have

$$u_n'(1) = -\int_0^1 f(r, u_n) r^{\alpha_n} \, dr \to 0 \quad \text{as } \alpha_n \to 1.$$

Since for any $0 \leqslant r \leqslant 1$ we have

$$r^\alpha u_n'(r) = u_n'(1) + \int_r^1 f(t, u_n) t^{\alpha_n} \, dt$$

we have

$$\sup_{r \in [0,1]} |r^\alpha u_n'(r)| \to 0 \quad \text{as } \alpha_n \to 1.$$

This shows for any $0 < r_0 \leqslant 1$,

$$\sup_{r_0 \leqslant r \leqslant 1} |u_n'(r)| \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.70}$$

This in turn implies

$$\sup_{r_0 \leqslant r \leqslant 1} |u_n(r)| \leqslant \int_{r_0}^1 |u_n'(t)| \, dt \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.71}$$

From (ii) of definition (2.1) and (3.33) we have

$$\tfrac{1}{2}\delta_0^2 u_n'(\delta_0)^2 = (1+\alpha_n)\int_0^{\delta_0} F(r,u_n) r^{\alpha_n}\,\mathrm{d}r + \int_0^{\delta_0} \frac{\partial F}{\partial r}(r,u_n) r^{1+\alpha_n}\,\mathrm{d}r$$

$$+\frac{(1-\alpha)}{2}\int_0^{\delta_0} u_n'(r)^2 r^{\alpha_n}\,\mathrm{d}r - \delta_0^{1-\alpha_n} F(\delta_0, u_n(\delta_0))$$

$$\geqslant (1+\alpha)\int_0^{\delta_0} F(r,u_n) r^{\alpha_n}\,\mathrm{d}r - \delta_0^{1+\alpha_n} F(\delta_0, u_n(\delta_0)) \tag{3.72}$$

Hence by (3.70) and (3.72) we have

$$\int_0^{\delta_0} F(r,u_n) r^{\alpha_n}\,\mathrm{d}r \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.73}$$

From (3.71) and by dominated convergence theorem

$$\int_{\delta_0}^{1} F(r,u_n) r^{\alpha_n}\,\mathrm{d}r \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.74}$$

Combining (3.73) and (3.74) we have

$$\int_0^{1} F(r,u_n) r^{\alpha_n}\,\mathrm{d}r \to 0 \quad \text{as } \alpha_n \to 1. \tag{3.75}$$

Let N_0 be such that for all $n \geqslant N_0$,

$$\int_0^{1} F(r,u_n) r^{\alpha_n}\,\mathrm{d}r < \frac{\eta}{2}. \tag{3.76}$$

From (ii) and (iii) of (3.68) and (3.76)

$$\tfrac{1}{2}\|u_n\|_{\alpha_n}^2 = \bar{I}_{\alpha_n}(u_n) + \int_0^1 F(r,u_n) r^{\alpha_n}\,\mathrm{d}r$$

$$< \left(l_{\alpha_n} + \frac{\eta}{2}\right) + \frac{\eta}{2} = (l_{\alpha_n} + \eta)$$

$$\leqslant \frac{\gamma}{b}.$$

Hence

$$|u_n(r)|^2 \leqslant \|u\|_1^2 \log\frac{1}{r}$$

$$< 2(l_{\alpha_n} + \eta)\log\frac{1}{r}$$

$$\leqslant \frac{2\gamma}{b}\log\frac{1}{r}. \tag{3.77}$$

From (3.32), (3.70) and (3.77) we have

$$
\begin{aligned}
u_n(0) &= \int_0^1 t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right) f(t,u_n)\,\mathrm{d}t \\
&= \int_0^{\delta_1} t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right) f(t,u_n)\,\mathrm{d}t + \int_{\delta_1}^1 t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right) f(t,u_n)\,\mathrm{d}t \\
&\leqslant M\int_0^{\delta_1} t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\exp(bu_n^2)\,\mathrm{d}t + M_1 \\
&\leqslant M\int_0^{\delta_1} t^{\alpha_n}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\exp\left(2\gamma\log\frac{1}{t}\right)\mathrm{d}t + M_1 \\
&\leqslant M\int_0^{\delta_1} t^{\alpha_n-2\gamma}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\mathrm{d}t + M_1
\end{aligned}
\tag{3.78}
$$

Now choose $\varepsilon > 0$ such that

$$\alpha_n > 2\gamma - 1 + \varepsilon \quad \text{for all } n \text{, large.}$$

Then from (3.61) and (3.78) we have

$$
\begin{aligned}
u_n(0) &\leqslant M\int_0^{\delta_1} t^{\alpha_n-2\gamma-\varepsilon/2}t^{\varepsilon/2}\left(\frac{1-t^{1-\alpha_n}}{1-\alpha_n}\right)\mathrm{d}t + M_1 \\
&\leqslant \frac{2M}{\varepsilon}\frac{1}{\left(\alpha_n-2\gamma+1-\frac{\varepsilon}{2}\right)} + M_2 \leqslant \frac{4M}{\varepsilon^2} + M_1.
\end{aligned}
\tag{3.79}
$$

Hence

$$\|u_n\|_\infty = u_n(0) \leqslant \frac{4M}{\varepsilon^2} + M_1.$$

Since u_n is in $\sum_{\alpha_n}$ and $\{\|u_n\|_\infty\}$ is bounded and hence u_n converges strongly in $C[0,1]$ and in H_0^1 to a function u. From (3.71) $u_n(r) \to 0$ as $\alpha_n \to \infty$ for every $r \neq 0$, we have $u \equiv 0$ and hence $u_n(0) \to 0$. Now choose N large such that $\|u_n\|_\infty \leqslant t_0$ for all $n \geqslant N$. From (iii) of Definition (2.1) we have

$$
\begin{aligned}
\lambda_{\alpha_n}\int_0^1 u_n\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r &= -\int_0^1 (r^{\alpha_n}\phi'_{\alpha_n})u_n\,\mathrm{d}r \\
&= -\int_0^1 (r^{\alpha_n}u'_n)'\phi_{\alpha_n}\,\mathrm{d}r \\
\lambda_{\alpha_n}\int_0^1 u_n\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r &= -\int_0^1 (r^{\alpha_n}u'_n)'\phi_{\alpha_n}\,\mathrm{d}r \\
&= \int_0^1 f(r,u_n)\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r \\
&< \lambda_{\alpha_n}\int_0^1 u_n\phi_{\alpha_n}r^{\alpha_n}\,\mathrm{d}r.
\end{aligned}
$$

and hence a contradiction. This proves the claim. Hence by going to a subsequence, we assume that

$$\lim_{\alpha_n \to 1} u_n'(1) \neq 0$$
$$l_{\alpha_n} \leqslant \bar{I}_{\alpha_n}(u_n) < 2l_{\alpha_n} \leqslant 4m^2. \tag{3.80}$$

Now

$$\begin{aligned} 4m^2 &\geqslant \frac{1}{2}\int_0^1 [f(r,u_n)u_n - 2F(r,u_n)]r^{\alpha_n}\,dr \\ &= \frac{1}{2}\int_0^1 [f(r,u_n)u_n - \beta F(r,u_n)]r^{\alpha_n}\,dr + \frac{\beta-2}{2}\int_0^1 F(r,u_n)r^{\alpha_n}\,dr \\ &\geqslant M_1 + \left(\frac{\beta-2}{2}\right)\int_0^1 F(r,u_n)r^{\alpha_n}\,dr \end{aligned}$$

where M_1 is constant independent of n. Hence $\exists M_2 > 0$ such that

$$\int_0^1 F(r,u_n)r^{\alpha_n}\,dr \leqslant M_2$$
$$\tfrac{1}{2}\|u_n\|_{\alpha_n}^2 = \bar{I}_{\alpha_n}(u_n) + \int_0^1 F(r,u_n)r^{\alpha_n}\,dr \leqslant 4m^2 + M_2. \tag{3.81}$$

Hence $\{\|u_n\|_{\alpha_n}\}$ is uniformly bounded. Hence from (3.80) and Lemma (3.6), u_n converges weakly to a non-zero solution u of (1.2).

From condition (i) of Theorem (2.1), we have for every $1 \leqslant p < \infty$, $f(u) \in L^p(D)$ (see Moser [6]). Hence by regularity of elliptic operators, $u \in w^{2,p}(D)$ and hence by Sobolev imbedding u is in $C^1(\bar{D})'$ and hence in $C^2(\bar{D})$. This proves the result.

Remark 3.1. From the proof of Theorem (2.1) it follows that if $m > 0$ is satisfying (2.11), then from Lemma (3.2) $l_\alpha \leqslant 2m^2$ and hence $l \leqslant 2m^2$. Therefore if $2m^2 b < 1$ implies $lb < 1$. This proves the criterion (2.10).

Proof of Theorem 2.2. From Lemma (3.2) there exists $\alpha_0 < 1$ such that $\sum_\alpha$ is non-empty and $\{a_\alpha\}$ is bounded for $\alpha_0 \leqslant \alpha < 1$. Lemma (3.3) gives (2.15).

Case (1). Let f be super critical and $\overline{\lim}_{\alpha \to 1} a_\alpha = a \neq 0$. Then from Lemma (3.4) we have

$$\sup_{\|w\|_1 \leqslant 1} \int_0^1 f(r, aw)wr\,dr \leqslant a.$$

contradicting the fact that f is super critical. Hence $a = 0$.

Case 2. If f is critical, let $a = \overline{\lim}_{\alpha \to 1} a_\alpha$. Then from (3.30) it follows that

$$\sup_{\|w\|_1 = 1} \int f(r, aw)wr\,dr \leqslant a$$

and from Lemma (3.8),

$$\frac{a^2}{2} b(0) < 1. \tag{3.82}$$

Now choose an ε and δ positive such that

$$\begin{aligned} &\text{(i)}\ \ f(r,t) \leqslant M \exp\left[(b(0)+\varepsilon)t^2\right] \quad \text{for all } (r,t) \in Q_\delta. \\ &\text{(ii)}\ \ \frac{a^2}{2}(b(0)+\varepsilon) < 1. \end{aligned} \tag{3.83}$$

Such a choice is possible because of (3.82) and the condition that f is critical.

Since $a_\alpha^2/2 = l_\alpha$, and hence f satisfies (2.10) of Theorem (2.1) with b replaced by $(b(0)+\varepsilon)$ and hence there exists a sequence u_n in $\sum_{\alpha_n}$ and a weak solution u of (1.2) such that

$$\begin{aligned} &\text{(iii)}\ \ I_{\alpha_n}(u_n) \to \frac{a^2}{2} \quad \text{as } \alpha_n \to 1 \\ &\text{(iv)}\ \ u_n \to u \quad \text{in } H_0^1. \\ &\text{(v)}\ \ \lim_{n\to\infty} \int_0^1 F(r,u_n) r^n \, dr = \int_0^1 F(r,u) r \, dr. \end{aligned} \tag{3.84}$$

In fact (iii) follows from Lemma (3.6). From weak lower semicontinuity of the norm we have

$$\|u\|_1^2 \leqslant \underline{\lim}_{\alpha_n \to 1} \|u_n\|_{\alpha_n}.$$

and hence from (iii) we have

$$I_1(u) \leqslant \underline{\lim}_{\alpha_n \to 1} I_{\alpha_n}(u_n) = \frac{a^2}{2}. \tag{3.85}$$

Let w be in B_{01}. Choose γ_α such that

$$\|w\|_\alpha^2 = \frac{1}{\gamma_\alpha} \int_0^1 f(r, \gamma_\alpha w) w r^\alpha \, dr.$$

Such a γ_α exists and $\lim_{\alpha\to 1} \gamma_\alpha = \gamma_1$ exists and is $\leqslant 1$ because w is in B_{01} and $\gamma_\alpha w$ is in B_α. Hence

$$\frac{a_\alpha^2}{2} \leqslant I(\gamma_\alpha w).$$

Taking the $\overline{\lim}$ as $\alpha \to 1$, we get

$$\frac{a^2}{2} \leqslant I_1(\gamma w) \leqslant I_1(w)$$

This implies

$$\frac{a^2}{2} \leqslant \inf_{B_{01}} I_1. \tag{3.86}$$

From Lemma (3.10), (3.85) and (3.86) and using the fact that u is in B_1^*, we get

$$I_1(u) = \frac{a^2}{2} = \inf_{B_{01}} I_1$$

and $a \neq 0$ because $u \not\equiv 0$. This proves Theorem (2.2).

Remark 3.2. Suppose $f(r,t) \leqslant 0$ for $r \in [0,1]$ and $0 \leqslant t \leqslant t_0$ and satisfying all other hypothesis on f, then also the Theorems (2.1) and (2.2) are valid.

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References

[1] Adimurthi, Infinite number of solutions of semilinear Dirichlet problem with critical growths in the unit disc in $\mathbb{R}^2$ (in Preparation)
[2] Adimurthi, Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the n-Laplacian (preprint)
[3] Atkinson F V and Peletier L A, Ground state and Dirichlet problems for $-\Delta u = f(u)$ in $\mathbb{R}^2$, *Arch. Ration. Mech. Anal.* **96** (1986) 147–165
[4] Brezis H and Nirenberg L, Positive solutions of nonlinear elliptic equations involving critical exponents, *Comm. Pure Appl. Math.* **36** (1983) 437–477
[5] Carleson L and Chang A, On the existence of an extremal function for an inequality of J Moser, *Bull. Sc. Math. 2° Serie* **110** (1986) 113–127
[6] Moser J, A sharp form of an inequality by N Trudinger, *Indiana Univ. Math. J.* **20** (1971) 1077–1092
[7] Rabinowitz P H, Minimax methods in critical point theory with applications to differential equations, *Regional Conference Series in Mathematics* (AMS) Vol. 65
[8] Zeev-Nehari, On a class of nonlinear second order differential equations, *Trans. AMS* **95** (1960) 101–123

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A softer, stronger Lidskii theorem

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Abstract. We provide a new approach to Lidskii's theorem relating the eigenvalues of the difference $A-B$ of two self-adjoint matrices to the eigenvalues of A and B respectively. This approach combines our earlier work on the spectral matching of matrices joined by a normal path with some familiar techniques of functional analysis. It is based, therefore, on general principles and has the additional advantage of extending Lidskii's result to certain pairs of normal matrices. We are also able to treat some related results on spectral variation stemming from the work of Sunder, Halmos and Bouldin.

Keywords. Lidskii theorem; spectral variation; functional analysis; normal path inequality.

1. Introduction

To discuss the classical Lidskii theorem we first recall the notion of majorization. For real or complex vectors v, w ($\in\mathbb{C}^n$), we define majorization (of v by w), written $v \ll w$, to mean that v is a convex combination of the permutations σw of w (σw denotes the vector obtained from w by permuting the components according to the permutation σ of $\{1,\ldots,n\}$). For our purposes it will be convenient to define also "soft" majorization (of v by w), written $v \ll_s w$, to mean that $v = \Sigma z_k \sigma_k w$ (finite sum) where the $\sigma_k w$ are permutations of w and the z_k are complex numbers such that $\Sigma |z_k| \leqslant 1$. It is clear that $\Sigma_1^n v_j = \Sigma_1^n w_j$ when $v \ll w$; on the other hand if these quantities are equal and non-zero then $v \ll w$ follows from $v \ll_s w$, because Σz_k must be 1.

For any n-by-n matrix T, we write $\operatorname{Eig} T$ to indicate the n-vector of eigenvalues of T, including multiplicity and ordered arbitrarily. In other contexts $\operatorname{Eig} T$ may stand for the related diagonal matrix with eigenvalues of T on its diagonal. Lidskii's theorem (see, for example, Kato [13, §6.5 of chap. 2]) says that if A and B are self-adjoint, α_* denotes the version of $\operatorname{Eig} A$ with eigenvalues in decreasing order, and β_* is the same for B, then

$$\alpha_* - \beta_* \ll \operatorname{Eig}(A-B). \tag{1.1}$$

For the early history of this theorem see Bhatia [5, §9 and the notes and references for chap. III]. Many of the earlier proofs require smoothness results on the eigenvalues of the intermediate matrices $(1-t)A + tB$; a recent proof due to Hiai and Nakamura [12] is not encumbered in that way, but rests on a rather intricate interpolation method.

Our approach stresses the relation between (1.1) and norm inequalities. It has long been recognized that (1.1) implies

$$\mu(\operatorname{diag}(\alpha_*) - \operatorname{diag}(\beta_*)) \leqslant \mu(A - B), \tag{1.2}$$

for every strongly unitarily invariant (sui) norm μ (as in [8], we say that a norm μ on the space of matrices is sui if it satisfies $\mu(UTV) = \mu(T)$ for every matrix T and unitary U and V). This implication is traditionally based on the well-known relation between sui norms, symmetric gauge functions and majorization. In fact, however, a comment of Ando (see [1, Theorem 7.1]) makes it clear that we may deduce (1.2) directly for the more general weakly unitarily invariant (wui) norms. A wui norm τ is a norm on the space of matrices that satisfies $\tau(U^* TU) = \tau(T)$ for every T and unitary U. Ando pointed out that if S and T are self-adjoint and $\operatorname{Eig} S \ll \operatorname{Eig} T$ then there are unitary U_k such that S is a convex combination of the $U_k^* T U_k$. It follows that

$$\tau(\operatorname{diag}(\alpha_*) - \operatorname{diag}(\beta_*)) \leqslant \tau(A - B), \tag{1.3}$$

for every wui norm τ.

Our strategy will be to prove (1.3), and more, using our general "normal path inequality" (see §2), then reverse the implications discussed in the last paragraph (via "soft" functional analysis – see §4) to obtain (1.1), and more.

We shall use $\mathbb{N}(n)$ to denote the set of normal operators on complex n-space $\mathbb{C}^n$.

2. The normal path inequality

Given any wui norm τ and matrices T and S, we define the τ-spectral distance between T and S, denoted by $\tau(\operatorname{Eig} T, \operatorname{Eig} S)$, by setting

$$\tau(\operatorname{Eig} T, \operatorname{Eig} S) = \min\{\tau(\operatorname{Eig} T - \operatorname{Eig} S)\}, \tag{2.1}$$

where the minimum is taken over all orderings of the diagonal matrix $\operatorname{Eig} S$; because τ is wui (and permutation matrices are unitary), $\tau(\operatorname{Eig} T - \operatorname{Eig} S)$ depends only on the relative ordering of the two diagonal matrices and the spectral distance is a pseudometric.

In [8] we showed that the spectral distance between any two normal operators, measured by a fixed wui norm τ, is bounded by the τ-length of any normal path joining the operators. For our present purposes it is important to observe that, for a fixed path, the matching of eigenvalues may be determined by "following the path" so that the matching can be the same for all wui norms τ. To see this we first recall a key lemma from [8].

Lemma 1 (*Proposition 5.2 of* [8]). *For fixed* wui *norm τ, normal N_0, and $\varepsilon > 0$,*

$$\tau(\operatorname{Eig} N_1, \operatorname{Eig} N_0) \leqslant (1 + \varepsilon)\tau(N_1 - N_0) \tag{2.2}$$

whenever N_1 is normal and sufficiently close to N_0.

Remark. In [8] we discussed certain situations where $\varepsilon = 0$ is appropriate in the foregoing lemma, relating this phenomenon to some work of Halmos and Bouldin; in

§7 below, we identify a broad class of norms exhibiting this behaviour.

In the following proposition $\tau(N(\cdot))$ denotes the arc-length of a curve $N(t)$ (t in some parametric interval) measured via the metric induced by the (wui) norm τ. In cases of interest this will be finite (i.e. the curve will be τ-rectifiable). In view of the proposition it is important to determine how short this arc-length can be made by a suitable choice of normal path; in [7] there are results of this type.

PROPOSITION 2

If $N(\cdot)$ is a path defined on $[0,1]$ and with values in $\mathbb{N}(n)$ then there is a fixed ordering of Eig $N(0)$ *and* Eig $N(1)$ *such that, for all* wui *norms* τ

$$\tau(\operatorname{Eig} N(1) - \operatorname{Eig} N(0)) \leqslant \tau(N(\cdot)). \tag{2.3}$$

Proof. Since $N(t)$ varies continuously with t, standard results on spectral continuity (see, for example, [13, § 5.2 of chap. 2]) ensure that there are continuous functions $\mu_k(t)$ such that $\{\mu_1(t), \ldots, \mu_n(t)\}$ is the spectrum (with multiplicity) of $N(t)$. We claim that, for any such system of functions, (2.3) is satisfied when we choose $\mu_1(1), \ldots, \mu_n(1)$ as the ordering for the eigenvalues in Eig $N(1)$ and $\mu_1(0), \ldots, \mu_n(0)$ as the ordering for Eig $N(0)$.

Fix a wui norm τ and $\varepsilon > 0$, and let G be the set of $t \in [0,1]$ such that $\tau(D(t) - D(0)) \leqslant (1+\varepsilon)\tau(N[0,t])$, where $D(t)$ denotes the diagonal matrix with $\mu_k(t)$ as the kth diagonal entry and $N[s,t]$ denotes the part of the path $N(\cdot)$ defined on $[s,t]$. By continuity it is clear that G includes its supremum (maximum) g. We wish to show that $g = 1$. If this is not the case. Lemma 1 ensures that for $g' > g$ and sufficiently close to g there is an ordering of Eig $N(g')$ such that

$$\tau(D(g) - \operatorname{Eig} N(g')) \leqslant (1+\varepsilon)\tau(N(g) - N(g')). \tag{2.4}$$

Let $\alpha_1, \ldots, \alpha_m$ be the distinct eigenvalues of $D(g)$. There is some permutation σ such that Eig $N(g') = \sigma D(g')$ (or, more properly, the unitary similarity corresponding to σ applied to $D(g')$). Since both sides of (2.4) tend to 0 as g' approaches g, the continuity of the functions $\mu_k(\cdot)$ forces $\tau(\operatorname{Eig} N(g') - D(g'))$ to be small relative to the minimum distance between the α_k, for g' sufficiently close to g. Thus, if $\sigma(i) = j$, both i and j must index eigenvalues of $D(g')$ close to the same α_k; in other words, $\sigma D(g) = D(g)$ $(= \sigma^{-1} D(g))$. Since τ is wui,

$$\tau(D(g) - D(g')) = \tau(\sigma^{-1} D(g) - D(g')) = \tau(D(g) - \sigma D(g')), \tag{2.5}$$

so that (2.4) yields

$$\tau(D(g) - D(g')) \leqslant (1+\varepsilon)\tau(N(g) - N(g')). \tag{2.6}$$

Combining this inequality with

$$\tau(D(0) - D(g)) \leqslant (1+\varepsilon)\tau(N[0,g]), \tag{2.7}$$

we obtain

$$\tau(D(0) - D(g')) \leqslant (1+\varepsilon)\tau(N[0,g']), \tag{2.8}$$

so that $g' \in G$; this contradiction shows that $g = 1$. Since the inequality

$$\tau(D(0) - D(1)) \leqslant (1 + \varepsilon)\tau(N(\cdot)) \tag{2.9}$$

holds for all $\varepsilon > 0$ and all wui norms τ, our claim is established.

q.e.d.

Remark. The significance of the fixed matching of eigenvalues in the result above should perhaps be stressed. In general the matching that minimizes $\tau(\mathrm{Eig}\,T - \mathrm{Eig}\,S)$ depends on τ; examples of this phenomenon (where T is self-adjoint and S skew-adjoint) are discussed in [2].

3. A Lidskii theorem for normal operators

We have already discussed in § 1 the general strategy for proving the following theorem; the details of the proof may be found in § 5 below.

Theorem 3. *For any $T, S \in \mathbb{N}(n)$ such that $T - S$ is also normal, there is an ordering of Eig T and Eig S such that*

$$\mathrm{Eig}\,T - \mathrm{Eig}\,S \ll \mathrm{Eig}\,(T - S). \tag{3.1}$$

The restrictions imposed by the hypotheses of this theorem are discussed in § 6 *below.*

COROLLARY 4 (*Lidskii's theorem*)

If A and B are self-adjoint then

$$\alpha_* - \beta_* \ll \mathrm{Eig}\,(A - B), \tag{3.2}$$

where α_* is the version of Eig A *with the eigenvalues arranged in decreasing order, and β_* has the same relationship to B.*

Proof. The difference $A - B$ is again self-adjoint, hence normal. It is a well-known fact (see [15, chap. 6, § A]) about the partial ordering $\ll$ that for any real vectors α and $\beta, \alpha_* - \beta_* \ll \alpha - \beta$.

q.e.d.

The following result was proved by Sunder [17] for the case of sui norms and by a different, rather intricate, argument.

COROLLARY 5

If $T, S, T - S \in \mathbb{N}(n)$ then for each wui *norm τ there is an ordering of* Eig T *and* Eig S *(which may depend on τ) such that*

$$\tau(T - S) \leqslant \tau(\mathrm{Eig}\,T - \mathrm{Eig}\,S). \tag{3.3}$$

Proof. Apply the theorem with T replaced by $T - S$ and S by $-S$ to get an ordering

such that

$$\operatorname{Eig}(T-S)+\operatorname{Eig}(S)=\Sigma t_k\sigma_k(\operatorname{Eig} T), \tag{3.4}$$

where the σ_k are permutations and $t_k \geqslant 0$ with $\Sigma t_k = 1$. Then

$$\operatorname{Eig}(T-S)=\Sigma t_k(\sigma_k(\operatorname{Eig} T)-\operatorname{Eig} S) \tag{3.5}$$

so that $\tau(T-S) \leqslant \Sigma t_k \tau(\sigma_k(\operatorname{Eig} T)-\operatorname{Eig} S) \leqslant \max_k \tau(\sigma_k(\operatorname{Eig} T)-\operatorname{Eig} S)$.

q.e.d.

Remark. For self-adjoint matrices the reasons for the inequality (3.3) may be identified more precisely. If A and B are self-adjoint, α_* is as in Corollary 4, and β^* puts $\operatorname{Eig} B$ in increasing order, then

$$\operatorname{Eig}(A-B) \ll \alpha_* - \beta^*, \tag{3.6}$$

as has been explicitly noted in [6]. By well-known properties of sui norms τ, it follows that

$$\tau(A-B) \leqslant \tau(\alpha_* - \beta^*); \tag{3.7}$$

actually, Ando's observation (see § 1) shows directly that (3.7) holds more generally for wui norms.

4. Software

PROPOSITION 6

For any operators X and Y on $\mathbb{C}^n$, the statements

$$\tau(X) \leqslant \tau(Y) \text{ for every wui norm } \tau \tag{4.1}$$

and

$$X = \Sigma z_k U_k^* Y U_k \text{ (finite sum) for some unitary operators } U_k \text{ and complex } z_k \text{ with } \Sigma|z_k| \leqslant 1 \tag{4.2}$$

are equivalent.

Proof. It is immediate from the definition of wui norms that (4.1) follows from (4.2). For the converse, suppose that (4.2) fails, i.e. X is not in the set K consisting of all finite sums of the type described in (4.2). Then for some $\varepsilon > 0$ X is also outside the set K_ε defined by $K_\varepsilon = K + \{Z : \|Z\| \leqslant \varepsilon\}$. Let τ be the Minkowski functional corresponding to K_ε; K_ε is convex, absorbing, bounded, circled, and invariant under unitary similarities, so that τ is a wui norm. By construction $\tau(X) > 1 \geqslant \tau(Y)$.

q.e.d.

COROLLARY 7

For any operators X and Y on $\mathbb{C}^n$, $\tau(X) = \tau(Y)$ for all wui *norms τ exactly when $X = \exp(i\theta) U^* Y U$ for some unitary U and real θ.*

Proof. If $\tau(X)=\tau(Y)$ for all wui norms τ, Proposition 6 ensures that X may be expressed in the form (4.2). Collect together any linearly dependent summands (dependent $U_k^* Y U_k$ and $U_j^* Y U_j$ would have to differ by a factor $\exp(i\alpha)$). Certain wui norms are strictly convex (e.g. the Hilbert–Schmidt (or Frobenius) norm is sui and induces an Euclidian metric on the space of matrices). For such a norm we cannot have $\tau(X)=\tau(Y)$ unless there is only a single summand in the expression for X. Clearly we must also have $|z_1|=1$.

q.e.d.

Remark. This corollary makes it clear that wui norms can behave quite differently from the more familiar sui norms. For example, $\tau(T^*)$ may differ from $\tau(T)$ for a wui norm τ (even when T is normal). Let T be a unitary of dimension 3 or more. The eigenvalues of T^* are the complex conjugates of those for T so that one spectrum cannot be obtained from the other by a rotation of the unit circle (excluding certain very special geometries for the spectra). We cannot, therefore, have $T^*=\exp(i\theta)U^*TU$ as in the corollary, so there is some wui norm τ with different values at T and T^*. Another such phenomenon occurs with positive definite matrices. If τ is a sui norm it is not hard to see that $\tau(A)\geqslant\tau(B)$ whenever $A\geqslant B\geqslant 0$. This is not the case for wui norms, in general; consider the wui norm τ defined in terms of the numerical radius $w(T)$ and numerical range $W(T)$ (both invariant under unitary similarities) by $\tau(T)=w(T)+\operatorname{diam}(W(T))$. If the spectrum of B is more dispersed than that of A it can certainly happen that $\tau(B)$ exceeds $\tau(A)$.

Remark. We have recently seen the preprint by Li and Tsing [14], who also developed "software" similar to Proposition 6 and Corollary 7.

PROPOSITION 8

For any $N, M\in\mathbb{N}(n)$, the statements

$$\tau(N)\leqslant\tau(M) \quad \textit{for every wui norm } \tau \tag{4.3}$$

and

$$\operatorname{Eig} N \ll_s \operatorname{Eig} M \tag{4.4}$$

are equivalent.

Proof. Since N and M are normal, they are unitarily equivalent to the matrices $\operatorname{Eig} N$ and $\operatorname{Eig} M$ respectively. Thus Proposition 6 tells us that (4.3) is equivalent to

$$\operatorname{Eig} N=\Sigma z_k U_k^* \operatorname{Eig} M U_k \text{ (finite sum) for some unitary operators } U_k \text{ and complex } z_k \text{ with } \Sigma|z_k|\leqslant 1. \tag{4.5}$$

It is easy to check that the diagonal entries of $U^*\operatorname{diag}(\beta)U$, where β is any complex n-vector and U is a unitary matrix $[u_{ij}]$, are given by the vector $D\beta$ with $D=[|u_{ji}|^2]$. Since U is unitary, the corresponding D is doubly stochastic. Thus by equating diagonals in (4.5) we obtain

$$\operatorname{Eig} N=\Sigma z_k D_k \operatorname{Eig} M \text{ (finite sum) for some doubly stochastic matrices } D_k \text{ and complex } z_k \text{ with } \Sigma|z_k|\leqslant 1; \tag{4.6}$$

note that in (4.6) we regard $\operatorname{Eig} N$ and $\operatorname{Eig} M$ as vectors. By a well-known theorem of Garrett Birkhoff (see, e.g. [1] or [15]) each doubly stochastic matrix is a convex combination of permutation matrices so that (4.6) is equivalent to (4.4). On the other hand, (4.4) directly implies (4.5) with permutation matrices as the U_k.

q.e.d.

5. Proof (soft) of Theorem 3

It is easy to check that if $T, S, T-S \in \mathbb{N}(n)$ then the direct path $N(t) = T + t(S-T)$ lies entirely in $\mathbb{N}(n)$. Applying Proposition 2 to this path we see that there is an ordering for $\operatorname{Eig} T$ and $\operatorname{Eig} S$ such that $\tau(\operatorname{Eig} T - \operatorname{Eig} S) \leqslant \tau(T-S)$ $(=\tau(N(\cdot)))$ for all wui norms τ. Applying Proposition 8 with $N = \operatorname{Eig} T - \operatorname{Eig} S$ and $M = T - S$ we conclude that $\operatorname{Eig} T - \operatorname{Eig} S \ll_s \operatorname{Eig}(T-S)$. By wiggling (e.g. replacing T by $T + \varepsilon I$ for small ε) we many assume that T and S have different traces. Then since the components of the vectors $\operatorname{Eig} T - \operatorname{Eig} S$ and $\operatorname{Eig}(T-S)$ have the same non-zero sum, we must have majorization rather than soft majorization.

q.e.d.

6. The condition $T - S$ normal; relation to the classical Lidskii theorem

It does not seem easy to clarify the domain of Theorem 3, i.e. to understand the structure of those pairs T and S such that T, S, and $T - S \in \mathbb{N}(n)$. Certainly they include pairs of the form

$$T = \oplus(z_k A_k + w_k), \quad S = \oplus(z_k B_k + v_k), \tag{6.1}$$

where the space is decomposed into a finite sum of orthogonal subspaces H_k. The operators A_k, B_k are self-adjoint on H_k and z_k, w_k, v_k are complex scalars. Note that it would be possible to obtain Theorem 3 for pairs of the form (6.1) by repeated applications of the classical Lidskii theorem (for self-adjoints). In dimensions three and up, examples show that Theorem 3 applies to a wider class of pairs, so that we have a more substantial extension of Lidskii's result as well as a new approach. In the two-dimensional case, however, a calculation shows that T, S, and $T-S$ are normal only when a representation of the form (6.1) exists.

7. Spectral variation in Q-norms

Following the terminology in [3] we shall say that a sui norm τ is a Q-norm if there exists another sui norm τ' such that for every A we have

$$\tau(A) = [\tau'(|A|^2)]^{1/2}, \tag{7.1}$$

where $|A|^2 = A^*A$. A Schatten p-norm is a Q-norm iff $p \geqslant 2$. The class of Q-norms, however, includes other interesting norms as well. For instance, if $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant s_n(A)$ is an enumeration of the singular values of an n-by-n matrix A,

then for each $k = 1, 2, \ldots, n$ and $1 \leqslant p < \infty$ the expression

$$\|A\|_{k,p} = \left[\sum_{j=1}^{k} (s_j(A))^p \right]^{1/p} \tag{7.2}$$

defines a sui norm on matrices (see, e.g. [16]). If $p \geqslant 2$, then for each $k = 1, 2, \ldots, n$ the norm defined by (7.2) is a Q-norm. To see this simply note that for such p we have

$$\|A\|_{k,p} = (\| |A|^2 \|_{k,p/2})^{1/2} \tag{7.3}$$

so that the requirement (7.1) is satisfied. The Schatten p-norms are a special case of (7.2) for $k = n$.

In some recent work it has been observed that in the derivation of several operator inequalities the "quadratic character" (7.1) of Q-norms plays a special role. See [3], [2] for material directly related to our present discussion and [9], [4] for other operator inequalities involving Q-norms.

The purpose of this section is to point out the following; the result of Halmos [11] on spectral approximants of normal operators, established by him for the operator norm, was extended to the class of Schatten p-norms, $p \geqslant 2$, by Bouldin [10]; in [3] this result has been extended further to the wider class of Q-norms. Our results in [8, §4] made use of the above mentioned results of Halmos and Bouldin. So, by much the same arguments, they can now be extended to the class of Q-norms. In particular, we have the following result on spectral variation.

PROPOSITION 9

Let $A, B \in \mathbb{N}(n)$ and let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be the respective eigenvalues of A and B. Suppose there is a permutation σ such that

$$|\alpha_i - \beta_{\sigma(i)}| \leqslant |\alpha_i - \beta_{\sigma(j)}| \tag{7.4}$$

for all i, j. Then for every Q-norm $\|\cdot\|_Q$ we have

$$\|\operatorname{diag}(\alpha_i - \beta_{\sigma(i)}\|_Q \leqslant \|A - B\|_Q. \tag{7.5}$$

Proof. Follow the model in [8, §4].

As noted in [8, Proposition 4.2], if the normal matrices A and B are sufficiently close their eigenvalues do meet the condition (7.4). Thus when τ is a Q-norm, Lemma 1 in §2 can be strengthened; its conclusion is true even when $\varepsilon = 0$. The proof of Proposition 2 can therefore be simplified somewhat in the case of a Q-norm.

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References

[1] Ando T, Majorization, doubly stochastic matrices and comparison of eigenvalues (lecture notes, Hokkaido University, Sapporo 1982) (to appear in *Linear Algebra & Appl*)

[2] Ando T and Bhatia R, Eigenvalue inequalities associated with the Cartesian decomposition, *Linear Multilinear Algebra* **22** (1987) 133–147

[3] Bhatia R, Some inequalities for norm ideals. *Commun. Math. Phys.* **111** (1987) 33–39

[4] Bhatia R, Perturbation inequalities for the absolute value map in norm ideals of operators. *J. Operator Theory* **19** (1988) 129–136

[5] Bhatia R, Perturbation bounds for matrix eigenvalues, *Pitman Research Notes in Mathematics* No. 162. Longman Scientific & Technical, Essex UK (1987)

[6] Bhatia R, The distance between the eigenvalues of Hermitian matrices, *Proc. Am. Math. Soc.* **96** (1986) 41–42

[7] Bhatia R and Holbrook J A R, Short normal paths and spectral variation, *Proc. Am. Math. Soc.* **94** (1985) 377–382

[8] Bhatia R and Holbrook J A R, Unitary invariance and spectral variation, *Linear Algebra & Appl.* **95** (1987) 43–68

[9] Bhatia R and Holbrook J A R, On the Clarkson–McCarthy inequalities, *Math. Ann.* **281** (1988) 7–12

[10] Bouldin R, Best approximation of a normal operator in the Schatten p-norm, *Proc. Am. Math. Soc.* **80** (1980) 277–282

[11] Halmos P R, Spectral approximants of normal operators, *Proc. Edinburgh Math. Soc.* **19** (1974) 51–58

[12] Hiai F and Nakamura Y, Majorization for generalized s-numbers in semifinite von Neumann algebras, *Math. Z.* **195** (1987) 17–27

[13] Kato T, *A short introduction to perturbation theory for linear opertors* (Berlin: Springer-Verlag) (1982)

[14] Li C-K and Tsing N-K, Norms that are invariant under unitary similarities and the C-numerical radii *Linear & Multilinear Algebra* (to appear)

[15] Marshall A W and Olkin I, *Inequalities: Theory of majorization and its applications* (New York: Academic Press) (1979)

[16] Okubo K, Hölder-type norm inequalities for Schur products of matrices, *Linear Algebra & Appl.* **91** (1987) 13–28

[17] Sunder V S, On permutations, convex hulls and normal operators, *Linear Algebra & Appl.* **48** (1982) 403–411

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On unbounded subnormal operators

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Abstract. A minimal normal extension of unbounded subnormal operators is established and characterized and spectral inclusion theorem is proved. An inverse Cayley transform is constructed to obtain a closed unbounded subnormal operator from a bounded one. Two classes of unbounded subnormals viz analytic Toeplitz operators and Bergman operators are exhibited.

Keywords. Unbounded subnormal operator; Cayley transform; Toeplitz and Bergman operators; minimal normal extension.

1. Introduction

Recently there has been some interest in unbounded operators that admit normal extensions viz unbounded subnormal operators defined as follows:

DEFINITION 1.1

Let S be a linear operator (not necessarily bounded) defined in $D(S)$, a dense subspace of a Hilbert space H. S is called a *subnormal operator* if it admits a normal extension $(N, D(N), K)$ in the sense that there exists a Hilbert space K, containing H as a closed subspace (the norm induced by K on H is the given norm on H) and a normal operator N with domain $D(N)$ in K such that $Sh = Nh$ for all $h \in D(S)$.

These operators appear to have been introduced in [12] following Foias [4]. An operator could be subnormal internally admitting a normal extension in H; or it could admit a normal extension in a larger space. As is well known, a symmetric operator always admits a self-adjoint extension in a larger space, contrarily a formally normal operator may fail to be subnormal ([2], [11]). Recently Stochel and Szafraniec ([12], [13]) obtained a Halmos–Bram type characterization of unbounded subnormal operators.

Here we discuss the existence and characterization of minimal normal extension N of an unbounded subnormal S. This is followed by the spectral inclusion theorem $\sigma(N) \subset \sigma(S)$. In §3, we set up a Cayley transform between a bounded subnormal and an unbounded one. We also exhibit two large classes of unbounded subnormals viz Bergman operators and analytic Toeplitz operators.

Let us recall [16, Ex. 5.39 p. 127] that given an operator T with domain $D(T)$ in a Hilbert space H, a closed subspace M of H is *invariant under* T if $T(D(T) \cap M) \subset M$. M is *reducing under* T if $T(M \cap D(T)) \subset M$, $T(M^{\perp} \cap D(T)) \subset M^{\perp}$ and $D(T) =$

$[M \cap D(T)] + [M^{\perp} \cap D(T)]$. Note that restriction of a normal operator to a reducing subspace is normal.

2. Minimal normal extension

DEFINITION 2.1

A normal extension $(N, D(N), K)$ of a subnormal operator $(S, D(S), H)$ is a *minimal normal extension* (MNE) if for any normal extension $(N_1, D(N_1), K_1)$ of S, $S \subset N_1 \subset N$ and K_1 is reducing under N implies $K_1 = K$ and $N_1 = N$.

In [13, p. 51] a normal extension N in K of S ($SD(S) \subset D(S)$) is called 'minimal' if $D = \{N^{*j}N^i x : x \in D(S),\ i, j = 0, 1, 2, \ldots\}$ is linearly dense in K. The second half of the following theorem shows that it is in fact a MNE. The class of *C^{∞}-vectors* for an operator T in H is $C^{\infty}(T) = \bigcap_{n=1}^{\infty} D(T^n)$.

Theorem 2.2. (a) *A subnormal operator admits a minimal normal extension.*
(b) *Let S be a subnormal operator with dense domain $D(S)$ in a Hilbert space H. Let $(N, D(N), K)$ be a normal extension of S. Let D be the linear span of $\{N^{*i}N^j x: i, j = 1, 2, \ldots;\ x \varepsilon C^{\infty}(S)\}$.*

(i) *If D is dense in K, then N is a* MNE
(ii) *If N is a* MNE *and $D(N) = D + (D(N) \cap D^{\perp})$, then D is dense in K.*

Proof. (a) Let $\mathscr{E}$ be the class of all normal extensions $\alpha = (N_{\alpha}, D(N_{\alpha}), K_{\alpha})$ of a subnormal operator S in a Hilbert space H with domain $D(S)$. $\mathscr{E}$ is partially ordered by $\alpha \leqslant \beta = (N_{\beta}, D(N_{\beta}), K_{\beta})$ if $N_{\alpha} \subset N_{\beta}$ and K_{α} is a reducing subspace for N_{β}. Note that for $\alpha \leqslant \beta$, the restriction $N_{\beta}|_{K_{\alpha}}$ of N_{β} on K_{α} with domain $D(N_{\beta}|_{K_{\alpha}}) = K_{\alpha} \cap D(N_{\beta})$ is a normal operator in K_{α} which is an extension in K_{α} itself of the normal operator N_{α}. Since a normal operator is maximally normal [10, p. 350], $N_{\alpha} = N_{\beta}|_{K_{\alpha}}$ so that $D(N_{\alpha}) = K_{\alpha} \cap D(N_{\beta})$. We shall apply Zorn's lemma to $\mathscr{E}$.

Let $\mathscr{C}$ be a chain in $\mathscr{E}$. Let $K = \cap \{K_{\alpha} | \alpha \in \mathscr{C}\}$, $D = \cap \{D(N_{\alpha}) | \alpha \in \mathscr{C}\}$. For $\alpha \in \mathscr{C}$, let $P^{\alpha}_{K}: K_{\alpha} \to K$ and for $\gamma \leqslant \alpha$, $P^{\alpha}_{\gamma}: K_{\alpha} \to K_{\gamma}$ be orthogonal projections. Now, let $\alpha \in \mathscr{C}$ be fixed. Since $\mathscr{C}$ is a chain, $K = \cap \{K_{\gamma} | \gamma \leqslant \alpha, \gamma \in \mathscr{C}\}$ and $D = \cap \{D(N_{\gamma}) | \gamma \leqslant \alpha, \gamma \in \mathscr{C}\}$.

Claim. K is a reducing subspace for the normal operator N_{α}. For this, note that $P^{\alpha}_{K} = \mathrm{glb}\{P^{\alpha}_{\gamma} | \gamma \in \mathscr{C}\} = \mathrm{glb}\{P^{\alpha}_{\gamma} | \gamma \in \mathscr{C}, \gamma \leqslant \alpha\}$, as in [15, p. 124]. Now consider the weak bounded commutant of N_{α} viz $\{N_{\alpha}\}' = \{S \in B(K_{\alpha}) | SN_{\alpha} \subset N_{\alpha}S\}$, $B(K_{\alpha})$ denoting the set of all bounded linear operators on K_{α}. By Fuglede–Putnam theorem for unbounded normal operators [10, p. 365], $\{N_{\alpha}\}' = \{S \in B(K_{\alpha}) | SN_{\alpha} \subset N_{\alpha}S, SN^{*}_{\alpha} \subset N^{*}_{\alpha}S\} = \{N_{\alpha}, N^{*}_{\alpha}\}'$. Let E be the spectral measure for the bounded normal operator $(1 + N^{*}_{\alpha}N_{\alpha})^{-1}$. For $k = 0, 1, 2, \ldots$. let $w_0(0)$, $w_k = (1/k + 1, 1/k]$, and $N_{\alpha,k} = N_{\alpha}E(w_k)$ which are bounded normal operators. Then as shown in the proof of Theorem 2·1 in [8], $\{N_{\alpha}\}' = \{N_{\alpha,k} | k = 0, 1, 2, \ldots\}'$ (usual commutant in $B(K_{\alpha})$ of a collection of bounded operators) which is a von Neumann algebra. Now by [16, p. 128], reducing subspaces of N_{α} correspond (via usual way of range projections) to projections in $\{N_{\alpha}\}'$. Hence for $\gamma \leqslant \alpha$, $P^{\alpha}_{\gamma} \in \{N_{\alpha}\}'$. Since projections in a von Neumann algebra form a complete lattice [15, p. 124], $P^{\alpha}_{K} \in \{N_{\alpha}\}'$; and hence K is a reducing subspace for N_{α}.

Now for $\gamma \leqslant \alpha$, $P^{\alpha}_{K}(D(N_{\alpha})) = K \cap D(N_{\alpha})$ since $D(N_{\alpha}) = [K \cap D(N_{\alpha})] + [K^{\perp} \cap D(N_{\alpha})]$.

Hence $P_K^\alpha(D(N_\alpha)) \subset K_\gamma \cap D(N_\alpha) = D(N_\gamma)$. Thus $P_K^\alpha(D(N_\alpha)) \subset \cap\{D(N_\gamma) | \gamma \in \mathscr{C}, \gamma \leqslant \alpha\} = D$. This implies that D is dense in K. For, given $x \in D^\perp$ ($\perp$ in K), for all $y \in D(N_\alpha)$, $\langle x, y\rangle = \langle P_K^\alpha x, y\rangle = \langle x, P_K^\alpha y\rangle = 0$, hence $x = 0$. Define an operator N in K with domain $D(N) = D$ as $Nx = N_\alpha x$. Then N is a well defined closed operator. To show that N is normal, consider an operator C in K with domain $D(C) = D$ as $Cx = N_\alpha^* x$ (adjoint in K_α). Then $C \subset N^*$ (adjoint in K). Now given $x \in D(N^*N)$, the functional $y \in D \to \langle N^*x, N^*y\rangle = \langle Cx, Cy\rangle = \langle N_\alpha^* x, N_\alpha^* y\rangle = \langle N_\alpha x, N_\alpha y\rangle = \langle Nx, Ny\rangle$ is continuous on D as $Nx \in D(N^*)$. Thus $N^*x \in D(N^{**}) = D(N)$ as N is closed. Thus $x \in D(NN^*)$ and $D(N^*N) \subset D(NN^*)$. In fact, $N^*N \subset NN^*$; and so $N^*N = NN^*$ both being self-adjoint (as N is closed). (Note that normality of N also implies $N = N_\alpha|_K$.)

The normal extension $(N, D(N), K)$ is a lower bound of $\mathscr{C}$. Now Zorn's lemma completes the proof.

(b) (i) Let the linear span D of $\{N^{*j}N^i x | x \in C^\infty(S);\, i,j = 1,2,\dots.\}$ be dense in K. Let $(N_0, D(N_0), K_0)$ be a normal extension of S such that $(N_0, D(N_0), K_0) \leqslant (N, D(N), K)$ (partial order as in the proof of (a)). Let $x \in C^\infty(S)$. Then for all $i = 1,2,\dots, SC^\infty(S) \subset C^\infty(S)$ gives that $N^i x = S^i x \in C^\infty(S) \subset C^\infty(N) \cap K_0$. Now for any positive integer k, by the normality of N^k, $D(N^k) = D((N^k)^*) = D((N^*)^k)$ which implies that $C^\infty(N) = C^\infty(N^*)$. Thus $N^{*j}N^i$ are defined for all $i, j = 1,2,\dots$. Further, since K_0 is invariant under N^*, $N^{*j}N^i x \in K_0$. Thus $D \subset K_0$. Hence $K_0 = K$, $N_0 = N$ showing that N is MNE.

(ii) Let $(N, D(N), K)$ be a MNE of S satisfying the given condition. Let $K_0 = \bar{D}$ (closure in K). By definition of D, $N^*D \subset D$, $ND \subset D$. These give $N(D(N) \cap K_0^\perp) \subset K_0^\perp$, $N^*(D(N) \cap K_0^\perp) \subset K_0^\perp$. Further, the given condition is equivalent to $D(N) = [D(N) \cap K_0] + [D(N) \cap K_0^\perp]$. We show that $N(D(N) \cap K_0) \subset K_0$. Let $x \in D(N) \cap K_0$. Then for all $y \in D(N) \cap K_0^\perp$, $\langle Nx, y\rangle = \langle x, N^*y\rangle = 0$. As $D(N) \cap K_0^\perp$ is dense in $K_0^\perp$, $Nx \in K_0$. Thus K_0 is reducing for N. Then $N|_{K_0}$ is a normal extension of S contained in N. By the minimality of N, $K_0 = K$. This completes the proof of the theorem.

The following is a spectral inclusion theorem analogous to the one for bounded subnormal. Its proof is patterned along Halmos [5, p. 157].

Theorem 2.3. *Let S be a subnormal operator in a Hilbert space H with domain $D(S)$ and a minimal normal extension N. Then $\sigma(N) \subset \sigma(S)$.*

Proof. Let $\lambda \notin \sigma(S)$. Then $(\lambda - S)^{-1}$ is a bounded operator on H. We can assume $\lambda = 0$ and $\|S^{-1}\| = 1$. Now for $0 < \varepsilon < 1$, consider $E_\varepsilon = \{x \in C^\infty(N) |\ \|N^n x\| < \varepsilon^n \|x\|$ for $n = 1,2,\dots.\}$. For $x \in E_\varepsilon$, $y \in H$,

$$
\begin{aligned}
|\langle x, y\rangle| &= |\langle x, S^n S^{-n} y\rangle| \\
&= |\langle N^{*n} x, S^{-n} y\rangle| \\
&\leqslant \varepsilon^n \|x\|\,\|y\| \text{ for all } n.
\end{aligned}
$$

As $\varepsilon < 1$, $\langle x, y\rangle = 0$. Thus $H \subset E_\varepsilon^\perp$ ($\perp$ in K). Let $N = \int z\, dE(z)$ be the spectral theorem for N. Then $E_\varepsilon = E(\Delta_\varepsilon)K$ where $\Delta_\varepsilon = \{z \in \mathbb{C} : |z| \leqslant \varepsilon\}$. Hence E_ε, and so $E_\varepsilon^\perp$ is a reducing subspace of N. Now $N|_{E_\varepsilon^\perp}$ being normal, the minimality of N implies that $E_\varepsilon = K$. Hence $E(\Delta_\varepsilon)K = E_\varepsilon = \{0\}$. Thus $\phi = \Delta_\varepsilon \cap \operatorname{supp} E = \Delta_\varepsilon \cap \sigma(N)$; and so $0 \notin \sigma(N)$.

Notice that, in above notations, bdry $\sigma(S) \subset \sigma_\pi(S) \subset \sigma_\pi(N) = \sigma(N)$ (σ_π denotes the approximate point spectrum) and component of $\mathbb{C}\backslash\sigma(N)$ is either contained in $\sigma(S)$ or is disjoint from $\sigma(S)$.

COROLLARY 1

Let S be a subnormal operator. Then

(i) $\sigma(S) \neq \phi$.
(ii) *S is bounded iff $\sigma(S)$ is bounded.*
(iii) *S is essentially self-adjoint iff $\sigma(S)$ is real.*

COROLLARY 2

A symmetric operator has nonempty spectrum.

Remarks 2·4. (i) Let S be an operator in a Hilbert space H. A vector $x \in C^\infty(S)$ is an *analytic vector for* S if there exists a $t > 0$ such that

$$\sum_{n=1}^{\infty} \frac{\|S^n x\| t^n}{n!} < \infty.$$

Let $A(S)$ be the collection of all analytic vectors for S. If S is subnormal admitting a normal extension N such that $D(S) = D(N) \cap H$, then $A(S)$ is dense in H. Indeed, in this case, $A(S) = A(N) \cap H$. Hence taking the orthogonal complement in K, $A(S)^\perp = H^\perp$ as $A(N)$ is dense in K.
(ii) A symmetric operator in H admitting a normal extension N in (possibly a larger space) K satisfying $D(S) = D(N) \cap H$ is essentially self-adjoint. For, in view of (i), the well-known Nelson theorem [16, p. 261] applies.
(iii) Normal extensions of an unbounded subnormal operator satisfying the above spectral inclusion (distinguished normal extensions) have been discussed recently in [6]. Thus a MNE is distinguished, though a distinguished extension need not be minimal. For example let N_1 be a MNE of S in K_1. Let N_2 be a normal operator in K_2 with $\sigma(N_2) \subset \sigma(N_1)$. Take $N = N_1 \oplus N_2$ a normal extension of S in $K_1 \oplus K_2 = K$. Then $N_0 = N|_{E(\sigma(S))K}$, (where E is the spectral measure of N) is distinguished normal extension as in [6] which is not minimal.
(iv) Ôta [7] showed that if T is a densely defined closed operator in a Hilbert space H such that $TD(T) \subset D(T^*)$, then T is bounded. This has the following implication.

PROPOSITION

Let S be a closed subnormal operator in H with dense domain D(S) such that SD(S) ⊂ D(S). Then S is bounded.

This follows from $D(S) \subset D(S^*)$ [14].

We are thankful to Prof. Ôta for bringing this to our notice.

(v) Ôta [7] has also another interesting result, viz if T is a densely defined closed operator in a Hilbert space H such that the range of T is contained in its domain and if T is unbounded, then the numerical range $W(T) = \{\langle Tx, x\rangle \mid x \in \mathcal{D}(T), \|x\| = 1\}$ is the entire complex plane. The following is an analogous result for spectrum.

PROPOSITION

Let T be a densely defined closed operator in a Hilbert space H such that $TD(T) \subset D(T)$. If $\sigma(T)$ is not the whole of complex plane, then T is bounded.

Proof. If $\lambda \notin \sigma(T)$, then $S = (T - \lambda 1)^{-1}$ is a bounded operator satisfying $S(T - \lambda 1) \subset (T - \lambda 1)S = 1$. Thus $(T - \lambda 1)D(T) \subset D(T) = H$. Closed graph theorem shows that T is bounded.

3. A Cayley transform

The problem of self-adjoint extension (within the space) of a symmetric operator is discussed via Cayley transform [15, Ch. 8] which provides a correspondence between certain partial isometries and symmetric operators that admit self-adjoint extensions. We extend this so as to associate an unbounded subnormal operator with a bounded one.

Theorem 3.1. *Let S be a bounded subnormal operator on H with a bounded normal extension N on K. If*

(i) $1 - N$ *is one-to-one and*
(ii) $1 \in \sigma(N)$, $\sigma(N) \backslash \{1\} \subset \{z \in \mathbb{C} : |z| < 1\}$

then $\psi(N)|_H$ is an unbounded closed subnormal operator where $\psi(N)$ is the normal operator in K defined via the spectral theorem by the function $\psi(z) = i(1 + z)(1 - z)^{-1}$.

Proof. Define N' in K with domain $D(N') = R(1 - N)$ by $N'x = i(1 + N)(1 - N)^{-1}x$. Then N' is densely defined.

Claim (a). $\overline{N'} = \psi(N)$.
For, given $x \in D(N')$, $(1 - N)y = x$, and so

$$\int |(1 + z)(1 - z)^{-1}|^2 \, \mathrm{d}E_{x,x} = \int |(1 + z)^2(1 - z)^{-2}|(1 - z)(1 - \bar{z}) \, \mathrm{d}E_{y,y} < \infty,$$

and for all $u \in K$

$$\langle N'x, u \rangle = i\langle (1 + N)y, u \rangle = i \int (1 + z)(1 - z)^{-1} \, \mathrm{d}E_{x,u}$$
$$= \langle \psi(N)x, u \rangle .$$

Hence $N' \subset \psi(N)$. As $\psi(N)$ is closed, $\overline{N'} \subset \psi(N)$. Now let $N_0 = N'^*|_{D(\psi(N^*)\psi(N))}$. Then $G(N_0)$ is dense in $G(N'^*)$, $G(\cdot)$ denoting the graph of the operator. Indeed, note that $G(N'^*)$ is closed in $K \times K$. Let $(u, N'^*u) \in G(N'^*)$, $(u, N'^*u) \perp G(N_0)$. Then for all $x \in D(\psi(N)^*\psi(N))$,

$$0 = \langle (u, N'^*u), (x, N'^*x) \rangle$$
$$= \langle u, x \rangle + \langle N'^*u, \psi(N)^*x \rangle \text{ (as } \psi(N)^* \subset N'^*)$$
$$= \langle u, x \rangle + \langle u, \psi(N)\psi(N)^*x \rangle . \qquad (\alpha)$$

Here we have used the following that can be easily verified.

Lemma. Let A and B be densely defined linear operators in a Hilbert space with B closed and $D(B) = D(B^*)$. If $A \subset B$, then for all $u \in D(A^*)$, $y \in D(B)$, $\langle A^*u, y\rangle = \langle u, By\rangle$.

Thus in (α), since $R(1 + \psi(N)^*\psi(N))$ is dense in K, $u = 0$. Then $G(N_0)$ is dense in $G(N'^*)$. Now, let $y \in D(N'^*)$. Then for some sequence (y_i) in $D(\psi(N)^*\psi(N))$, $y_i \to y$ and $N_0 y_i - N'^*y = N'^*y_i - N'^*y \to 0$. Since $\|\psi(N)y_i - \psi(N)y_j\| = \|\psi(N)^*y_i - \psi(N)^*y_j\| = \|N'^*y_i - N'^*y_j\|$, $(\psi(N)y_i)$ converges to some $u \in K$. Since $\psi(N)$ is closed, $(y, u) \in G(\psi(N))$, $y \in D(\psi(N))$. Thus $D(N'^*) \subset D(\psi(N)) = D(\psi(N)^*)$; hence $D(N'^*) = D(\psi(N)^*)$, $N'^* = \psi(N)$ and so $\overline{N'} = \psi(N)$.

Claim (b). H is invariant under $\psi(N)$ (and N'). For if, $x \in D(\psi(N)) \cap H$, $y \in H^\perp$, then since 1 is not an eigenvalue, $E(\{1\}) = 0$; and so

$$\begin{aligned}\langle \psi(N)x, y\rangle &= i\int_{\sigma(N)} (1+z)(1-z)^{-1}\mathrm{d}E_{x,y} \\ &= i\int_{\sigma(N)-\{1\}} (1+z)(1-z)^{-1}\,\mathrm{d}E_{x,y} \\ &= i\sum_k \int (1+z)z^k\,\mathrm{d}E_{x,y} \\ &= i\sum_k \langle (1+N)N^k x, y\rangle = 0\end{aligned}$$

as $(1 + N)\, N^k x \in H$. Thus $\psi(N)x \in H$. This establishes our claim (b).

It is easy to see that $\psi(N)|H$ with domain $D(\psi(N)|_H) = D(\psi(N)) \cap H$ is a closed operator.

Remark 3.2. Note that if $R(1 - S)$ (range of $(1 - S)$) is dense in H, then $S' = i(1 + S)(1 - S)^{-1}$ with domain $D(S') = R(1 - S)$ and $S_0 = N'|_H$ and hence are subnormals (not necessarily closed) in H.

4. Examples

4.1 *Unbounded analytic Toeplitz operators*

Let

$$U = \{z \in \mathbb{C} : |z| < 1\}, \quad \Gamma = \{z \in \mathbb{C} : |z| = 1\}.$$

Let ϕ be a measurable function on Γ and $D_\phi = \{f \in H^2(U) : \phi f \in L^2(\Gamma)\}$. Define T_ϕ in $H^2(U)$ with domain D_ϕ as $T_\phi f = P(\phi f)$, where $P: L^2(\Gamma) \to H^2(U)$ is the projection. The Toeplitz operator T_ϕ is an analytic Toeplitz operator if ϕ is analytic. Such a T_ϕ admits a normal extension M_ϕ with domain $D(M_\phi) = \{f \in L^2(\Gamma) : \phi f \in L^2(\Gamma)\}$, $M_\phi f = \phi f$. hus, in this case, if D_ϕ is dense in $H^2(U)$, then T_ϕ is subnormal. Note that it is nded iff ϕ is bounded. We exhibit below a class of function ϕ for which T_ϕ is a sed unbounded subnormal operator.

(i) $\phi(z) = (1-z)^{-1}$. Then $D_\phi = R(1-S)$ where S is the unilateral shift. Hence $D_\phi = \ker(1-S^*)^\perp = H^2(U)$. Also T_ϕ is closed. For, if $(f_n, T_\phi f_n) \to (f, g)$, then (identifying $H^2(U)$ with a closed subspace of $L^2(\Gamma)$), there exists a subsequence (f_{nk}) of (f_n) each of whose subsequence converges a.e. to f on Γ. Since $f_{n_k}(z)\,(z-1)^{-1} \to g \in L^2(\Gamma)$, $(z-1)g = f$ a.e. Hence $g = T_\phi f$ in $H^2(U)$.

(ii) A similar argument can be applied for $\phi(z) = (z-\lambda_1)^{-n_1}(z-\lambda_2)^{-n_2}\cdots(z-\lambda_k)^{-n_k}$ with $|\lambda_i| \geqslant 1$, $n_i = 1, 2, \ldots$.

(iii) As discussed in [6], functions $\phi \in H^2(U)$ define unbounded analytic Toeplitz operators.

Unbounded Teoplitz operators also arise quite naturally in representation of certain topological algebras by unbounded operators.

Consider Arens algebra [1] $L^w(\Gamma) = \bigcap_{1 \leqslant p < \infty} L^p(\Gamma) \neq L^\infty(\Gamma)$ with pointwise operations. It is a Frechet * algebra with the topology of L^p-convergence for each p, $1 \leqslant p < \infty$. The Hardy-Arens algebra $H^w(U) = \bigcap_{1 \leqslant p < \infty} H^p(U) \neq H^\infty(U)$ [9, Ch. 17, Ex. 10] can be regarded as a closed subalgebra of $L^w(\Gamma)$. For a $\phi \in H^w(U)$, D_ϕ is dense in $H^2(U)$ since $H^\infty(U) \subset D_\phi$ and $H^\infty(U)$ is dense in $H^2(U)$. In fact, as in (i) above, T_ϕ is closed. It is easily seen that $\phi \to T_\phi$ is a representation of $H^w(U)$ by unbounded subnormal operators in $H^2(U)$ which is the restriction of the unbounded * representation $\phi \to M_\phi$ of $L^w(\Gamma)$ into normal operators in $L^2(\Gamma)$.

4.2 *Unbounded Bergman operators*

Let G be a bounded domain in $\mathbb{C}$, For $1 \leqslant p < \infty$, consider the Bergman spaces $L^p_a(G) = \{f \in L^p(G) : f \text{ is analytic on } G\}$ with $\|\cdot\|_p$ norm. Let $L^w_a(G) = \bigcap_{1 \leqslant p < \infty} L^p_a(G)$. For $g \in L^w_a(G)$, define S_g in $L^2_a(g)$ with domain $D(S_g) = \{f \in L^2_a(G) : gf \in L^2_a(G)\}$ as $S_g f = gf$. Again S_g is densely defined if $L^\infty_a(G)$ is dense in $L^2_a(G)$, in particular, if G is a Caratheodory domain [3, Ch. 3] in which case $L^2_a(G) = P^2(G)$, the $L^2_a(G)$-closure of polynomials. In this way, one gets a large class of unbounded subnormals.

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References

[1] Arens R, The space L^w and convex topological rings, *Bull. Am. Math. Soc.* **52** (1946) 931–935.

[2] Coddington E A, Formally normal operators having no normal extension, *Can. J. Math.* **17** (1965) 1030–1040

[3] Conway J B, *Subnormal operators* (Boston, London, Melbourne: Pitman Adv. Publishing Programme) (1981)

[4] Foias C, Decompositions operateurs et vecteurs Propres I, *Rev. Roum. Math. Pures Appl.* **7** (1962) 241–281.

[5] Halmos P R, *A Hilbert space problem book* (Princeton D Van Nostrand) (1967)

[6] McDonald G and Sundberg, C, On the spectra of unbounded subnormal operators, *Can. J. Math.* **38** (1986) 1135–1148

[7] Ôta S, Closed linear operators with domain containing their ranges, *Proc. Edinburgh Math. Soc.* **27** (1984) 229–233
[8] Patel A B, A joint spectral theorem for unbounded operators, *J. Aust. Math. Soc. Ser. A* **34** (1983) 203–213
[9] Rudin W, *Real and complex analysis* (New York: McGraw Hill Publ. Co.) (1965)
[10] Rudin W, *Functional analysis* (New York: McGraw Hill Publ. Co.) (1976)
[11] Smüdgen K, Formally normal operator having no normal extension, *Proc. Am. Math. Soc.* **95** (1985) 503–504
[12] Stochel J and Szafranich F, Bounded vectors and formally normal operators, in *Dilation theory, Toeplitz operators and other topics* (ed). G. Arsene (Basel: Birkhauser, Verlag) (1983) 363–370
[13] Stochel J and Szafranich F, Normal extensions of unbounded operators, *J. Operator Theory* **14** (1985) 31–55
[14] Stochel J and Szafranich F, On normal extensions of unbounded operators II (preprint)
[15] Takesaki, M *Theory of operator algebras I* (New York: Springer-Verlag) (1976)
[16] Weidman J, *Linear operators in Hilbert spaces* (Graduate Text in Mathematics) (New York: Springer-Verlag) (1980)

'roc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 1, April 1989, pp. 93–101.

'hermohaline convection with cross-diffusion in an nisotropic porous medium

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Abstract. Using normal mode technique it has been shown that (i) values of the anisotropy parameter are important in deciding the mode of convection in a doubly diffusive fluid saturating a porous medium. (ii) Depending on the values of the Soret and Dufour parameters, an increase in anisotropy parameter either promotes or inhibits instability, (iii) cross-diffusion induces instability even in a potentially stable set-up and (iv) for certain values of the Dufour and Soret parameters there is a discontinuity in the critical thermal Rayleigh number, which disappears if the porous medium has horizontal isotropy.

Keywords. Anisotropy; cross-diffusion; porous medium; stability; thermohaline convection.

. Introduction

'he surface of the earth is a naturally occurring anisotropic porous medium. Hence he study of thermal convection in anisotropic porous media is of importance in eophysics, hydrology, oil extraction, etc. When both mass and heat transfer take lace simultaneously, they may interfere with each other producing cross-phenomena iving rise to Soret and Dufour effects. McDougall [6] and Knobloch [5] analysed his for a Newtonian fluid layer (i.e. in the absence of a porous medium). Tyvand 10] considered an anisotropic porous medium but ignored the cross-effects. Taslim nd Narusawa [9] made an analysis of cross-diffusion in a fluid saturated isotropic orous medium where the Dufour effect has been ignored in the results obtained. ut Ingle and Horne [4] reported a theoretical examination of Dufour effect in liquid iixtures.

The aim of the present paper is to study the effect of both cross-diffusion and nisotropy in permeability on double-diffusive convection in a fluid saturating an nisotropic porous medium. The normal mode technique of Chandrasekhar [1] is mployed in the analysis. Since anisotropy in permeability is comparatively of greater nportance than those of diffusivities (Neale [7]), anisotropy in diffusion properties ave been ignored in this preliminary analysis.

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2. Mathematical formulation

A quiescent layer of a Boussinesq double-diffusive fluid saturating an anisotropic porous medium of vertical thickness L extending to infinity in the horizontal direction is considered. The equations are in rectangular Cartesian system with the axes of co-ordinates in the directions of the principal axes of the anisotropic porous medium, one of which is assumed to be in the direction of the vertical z'-axis. The system has free permeable perfectly conducting boundaries at $z' = 0$ and $z' = L$ which are maintained at temperature T_0 and T_L respectively. The concentration of the diffusing component is held at C_0 and C_L at the boundaries. The Darcy's law is modified for anisotropy in permeability. The governing equations and the equation of state are

$$\nabla \cdot \mathbf{q} = 0 \tag{1}$$

$$0 = -\nabla p + \rho \mathbf{g} - \frac{\mu}{(k_x, k_y, k_z)} \mathbf{q} \tag{2}$$

$$\frac{\partial T}{\partial t'} + M \mathbf{q} \cdot \nabla T = D_{11} \nabla^2 T + D_{12} \nabla^2 C \tag{3}$$

$$\frac{\partial C}{\partial t'} + \mathbf{q} \cdot \nabla C = D_{21} \nabla^2 T + D_{22} \nabla^2 C \tag{4}$$

$$\rho = \bar{\rho}[1 - \alpha_T (T - \bar{T}) + s\alpha_c (C - \bar{C})] \tag{5}$$

where $\mathbf{q}$ is the seepage velocity, p the pressure, μ the fluid viscosity, ρ and $\bar{\rho}$ the densities at temperature T, $\bar{T}$ and concentration C, $\bar{C}$ where $\bar{T} = (T_0 + T_L)/2$, $\bar{C} = (C_0 + C_L)/2$; $\mathbf{g} = (0, 0, -g)$, g being the gravitational acceleration, (k_x, k_y, k_z) the permeabilities of the porous medium in the directions of the three axes, t' the time, $M = (\bar{\rho} c_p)_f / (\bar{\rho} c_p)_m$, c_p being the heat capacity and subscripts f and m standing for fluid and fluid-solid mixture respectively; D_{11} is the thermal diffusivity $= \lambda_m / (\bar{\rho} c_p)_m$, $\lambda_m = \phi \lambda_f + (1 - \phi)\, \lambda_s$ λ being the thermal conductivity and ϕ the porosity of the medium, subscript s stands for the solid matrix of the porous medium; D_{12} is the diffusion *thermo* or the Dufour coefficient; D_{21} is the thermal diffusion or the Soret coefficient; D_{22} is the mass or solutal diffusivity analogous to D_{11} for temperature; α_T and α_c are the coefficients of volume and mass expansivities respectively; $s = +1\ (-1)$ if the density of the diffusing component is greater (less) than that of the solvent. ∇ stands for $(\partial/\partial x')\hat{\imath} + (\partial/\partial y')\hat{\jmath} + (\partial/\partial z')\hat{k}$, $(\hat{\imath}, \hat{\jmath}, \hat{k})$ being the unit vectors in the directions of the axes of co-ordinates; ∇^2 stands for $(\partial^2/\partial x'^2) + (\partial^2/\partial y'^2) + (\partial^2/\partial z'^2)$. Imposing small perturbations on a basic quiescent state in which the gradients are present only in the vertical direction and linearising, the equations governing the perturbed quantities together with the boundary conditions using the normal modes

$$(\theta, S, W) = (\theta_0(z), S_0(z), W_0(z)) \exp(ia_x \times + ia_y y + \sigma t)$$

are

$$\left(\sigma - \frac{D^2 - a^2}{\Lambda_{11}}\right)\theta_0 - \frac{D^2 - a^2}{\Lambda_{12}} S_0 = M W_0 \tag{6}$$

$$\left(\sigma - \frac{D^2 - a^2}{\Lambda_{22}}\right) S_0 - \frac{D^2 - a^2}{\Lambda_{21}} \theta_0 = -W_0 \tag{7}$$

$$(D^2 - b)W_0 = -\frac{R_t}{\Lambda_{11}} b\theta_0 + \frac{sR_c}{\Lambda_{22}} bS_0 \tag{8}$$

$$W_0 = \theta_0 = S_0 = 0 \text{ at } z = 0, 1, \tag{9}$$

where the length, temperature, concentration, velocity and time are non-dimensionalised using L, $T_0 - T_L$, $C_L - C_0$, $\mu_L/\bar{\rho}k_z$ and $\bar{\rho}k_z/\mu$ respectively; D stands for $(\mathrm{d}/\mathrm{d}z)$; (θ, S, W) are the dimensionless temperature, concentration and velocity respectively; $a = (a_x^2 + a_y^2)^{1/2}$; $b = k_1 a_x^2 + k_2 a_y^2$; $k_1 = k_x/k_z$; $k_2 = k_y/k_z$; the thermal Rayleigh number modified for a porous medium

$$R_t = \frac{\alpha_T(T_0 - T_L)Lk_z g\bar{\rho}}{\mu D_{11}};$$

the solutal Rayleigh number modified for a porous medium

$$R_c = \frac{\alpha_c(C_L - C_0)Lk_z g\bar{\rho}}{\mu D_{22}}; \quad \Lambda_{11} = \frac{L^2}{k_z}\frac{\mu}{\bar{\rho}D_{11}} = \frac{L^2}{k_z}\mathrm{Pr},$$

Pr being the Prandtl number of the fluid;

$$\Lambda_{22} = \frac{L^2}{k_z}\frac{\mu}{\bar{\rho}D_{22}} = \frac{L^2}{k_z}\mathrm{Sc}, \text{ Sc being the Schmidt number;}$$

$$\Lambda_{12} = \frac{L^2}{k_z}\frac{\mu}{\bar{\rho}D_{12}}\frac{T_0 - T_L}{C_L - C_0}; \quad \Lambda_{21} = \frac{L^2}{k_z}\frac{\mu}{\bar{\rho}D_{21}}\frac{C_L - C_0}{T_0 - T_L}.$$

The quadratic equation in σ obtained by substituting $(\theta_0(z), S_0(z), W_0(z)) = (A_1, B_1, C_1) \sin n\pi z$ in (6)–(8) and eliminating A_1, B_1, C_1 for $M = 1$, $s = 1$, the least mode $n = 1$ is

$$\sigma^2(\pi^2 + b) + \sigma\left[(\pi^2 + b)\alpha^2\left(\frac{1}{\Lambda_{11}} + \frac{1}{\Lambda_{22}}\right) - b\left(\frac{R_t}{\Lambda_{11}} + \frac{R_c}{\Lambda_{22}}\right)\right]$$
$$+ \frac{(\pi^2 + b)\alpha^4}{\Lambda_{11}\Lambda_{22}}(1 - S_1 S_2) - \frac{b\alpha^2}{\Lambda_{11}\Lambda_{22}}[R_t(1 + S_1) + R_c(1 + S_2)] = 0, \tag{10}$$

where $\alpha^2 = \pi^2 + a^2$; the Soret parameter S_1

$$= \frac{\alpha_c D_{21}}{\alpha_T D_{22}} = \frac{R_c \Lambda_{11}}{R_t \Lambda_{21}};$$

the Dufour parameter S_2

$$= \frac{\alpha_T D_{12}}{\alpha_c D_{11}} = \frac{R_t \Lambda_{22}}{R_c \Lambda_{12}}.$$

The results for the onset of the finger (or stationary) and the diffusive (or oscillatory) instabilities are obtained by replacing σ with zero or $i\omega(\omega > 0)$ respectively where ω is the frequency.

2.1 *The finger instability*

Equation (10) for $\sigma = 0$ gives

$$R_t(1+S_1) = -R_c(1+S_2) + (1-S_1S_2)\frac{\alpha^2(\pi^2+b)}{b}.$$

Minimising R_t with respect to a_x, a_y the four possibilities are:

(i) $\quad a_x = 0, \quad a_y = 0.$

In a strict sense, the situation $a_x = 0 = a_y$ is meaningless because (8) and (9) imply that W_0 is identically zero and hence there can be no motion. But Nield [8] analysed the possibility of convection for zero critical wave number and concluded that disturbances can grow with time though relatively slowly and convection can therefore occur. His analysis is for a Newtonian fluid layer. Similar analysis could be carried out for the present problem.

(ii) $\quad a_x^2 + a_y^2 + \pi^2 = 0$ if $k_1 \neq k_2$.

This is not possible since a_x, a_y are real. For $k_1 = k_2 = \bar{k}$ (horizontal isotropy), a_c the critical wave number $= \pi(\bar{k})^{-1/4}$.

(iii) $\quad a_x = \pi k_1^{-1/4}, \quad a_y = 0.$

In this case $a_c = \pi k_1^{-1/4}$ and

$$R_{t_{\text{crit}}} = \frac{-(1+S_2)R_c + \pi^2(1-S_1S_2)(1+k_1^{-1/2})^2}{1+S_1} \tag{11}$$

(iv) $\quad a_x = 0, \quad a_y = \pi k_2^{-1/4}.$

This gives $a_c = \pi k_2^{-1/4}$ and

$$R_{t_{\text{crit}}} = \frac{-(1+S_2)R_c + \pi^2(1-S_1S_2)(1+k_2^{-1/2})^2}{1+S_1}. \tag{12}$$

The smaller of the two values of $R_{t_{\text{crit}}}$ obtained from (11) and (12) is the critical thermal Rayleigh number for the onset of finger instability and the corresponding value of a_c is the critical wave number. It is seen that if the horizontal permeability is more in one direction than the other, $k_1 > k_2$ say, (i.e. $k_x > k_y$) then the $R_{t_{\text{crit}}}$ given by (11) is smaller provided $1 - S_1S_2$ and $1 + S_1$ are of the same signs and the $R_{t_{\text{crit}}}$ given by (12) is smaller when $1 - S_1S_2$ and $1 + S_1$ are of the opposite signs.

2.2 *The diffusive instability*

Equation (10) for $\sigma = i\omega(\omega > 0)$, when the real and imaginary parts are equated to zero separately would lead to

$$R_t = -R_c\tau + \frac{(\pi^2+b)\alpha^2}{b}(1+\tau) \text{ and}$$

$$\omega^2 = \frac{-b\alpha^2}{\Lambda_{11}\Lambda_{22}(\pi^2 + b)} \left\{ R_c[(1 + S_2) - \tau(1 + S_1)] + \frac{(\pi^2 + b)\alpha^2}{b}[(1 + \tau)(1 + S_1) - (1 - S_1 S_2)] \right\}$$

where

$$\tau = \frac{\Lambda_{11}}{\Lambda_{22}} = \frac{\text{Pr}}{\text{Sc}}.$$

It is found that when diffusive instability exists, the critical wave number is the same as that for finger instability which together with the critical thermal Rayleigh number are given by

(i) $$R_{t_{\text{crit}}} = -R_c\tau + (1 + \tau)\pi^2(1 + k_1^{-1/2})^2; \quad a_c = \pi k_1^{-1/4} \tag{13}$$

or

(ii) $$R_{t_{\text{crit}}} = -R_c\tau + (1 + \tau)\pi^2(1 + k_2^{-1/2})^2; \quad a_c = \pi k_2^{-1/4} \tag{14}$$

The smaller of these two gives the $R_{t_{\text{crit}}}$ for the onset of diffusive instability.

The smaller of the $R_{t_{\text{crit}}}$ got from the finger and diffusive instabilities is the critical thermal Rayleigh number and the corresponding instability gives the mode of convection. A system for which $\omega^2 < 0$ can have only finger instability.

3. Discussion

It is seen from (11) to (14) that the critical thermal Rayleigh number and the critical wave number depend only on one of the two anisotropy parameters k_1, k_2 for either type of instability. The $R_{t_{\text{crit}}}$ for finger instability is independent of Pr and Sc while depending on S_1, S_2, the cross-diffusion effects; whereas $R_{t_{\text{crit}}}$ for diffusive instability is dependent on Pr Sc while independent of S_1, S_2, though the existence of the diffusive instability does depend on Pr, Sc, S_1 and S_2. The critical wave number is the same for both types of instability and is independent of Pr, Sc, S_1 and S_2. It may be noted that the diffusive instability exists for those values of R_c, Pr, Sc, S_1, S_2 and k_1 (or k_2) for which $\omega^2 > 0$. For both finger and diffusive instabilities there is a linear relationship between R_t and R_c for given values of the other parameters.

S_1, the Soret parameter, can take any value [3]. The values of S_1 and S_2 are such that $S_1 S_2 \neq 1$, because the temperature and concentration in the basic state are not unique otherwise. Since the authors have not come across the values of the Dufour parameter for a heat-solute pair though this is available for solute-solute pair [2], a broad-based discussion for some regions of S_1 and S_2 is given below.

Case (a). $S_1 + 1 = 0$.
At $S_1 = -1$, there is a singularity for the critical thermal Rayleigh number for finger instability or $R_{t_{\text{crit}}}$ becomes infinite implying that an infinite temperature difference is required to produce a non-oscillatory instability. But diffusive (or oscillatory) instability is possible ($\omega^2 > 0$) when $\{S_2 + 1 > 0$ and $R_c < \pi^2(1 + k_1^{-1/2})^2\}$ or $\{S_2 + 1 < 0$ and $R_c > \pi^2(1 + k_1^{-1/2})^2\}$ assuming $k_1 > k_2$. The corresponding $R_{t_{\text{crit}}}$ and a_c are given

by (13). It is seen from the value of $R_{t_{\text{crit}}}$ that an increase in the anisotropy parameter promotes diffusive instability i.e. if the horizontal permeability of the medium is more pronounced than the vertical permeability, the system becomes more unstable. When

$$S_2 + 1 > 0 \text{ and } \pi^2(1 + k_1^{-1/2})^2 < R_c < \pi^2(1 + k_2^{-1/2})^2 \tag{15}$$

diffusive instability sets in at a higher temperature gradient and a larger wave number with $R_{t_{\text{crit}}}$, a_c given by (14) implying that there may be a sudden shrinking of the cells and a discontinuity in $R_{t_{\text{crit}}}$ (as can be noticed from figure 1). This discontinuity disappears when the medium has horizontal isotropy. This happens when the system is salted above, subject to (15). But if R_c is further increased beyond $\pi^2(1 + k_2^{-1/2})^2$, it is found that convection through diffusive mode is not possible (because $\omega^2 < 0$).

According to the present analysis, the system, in this case, remains stable till an infinite temperature difference is attained. Perhaps a finite amplitude analysis may throw more light in understanding this situation. When $S_2 + 1 < 0$, a similar situation arises for $R_c < \pi^2(1 + k_1^{-1/2})^2$. For $R_c > \pi^2(1 + k_1^{-1/2})^2$ there is diffusive instability setting in with no discontinuity in $R_{t_{\text{crit}}}$.

Case (b). $S_1 + 1 \neq 0$.

The diffusive instability could exist ($\omega^2 > 0$) only if

$$R_c[(1 + S_2) - \tau(1 + S_1)] + \pi^2(1 + k_1^{-1/2})^2$$
$$[(1 + \tau)(1 + S_1) - (1 - S_1 S_2)] < 0 \tag{16}$$

assuming $k_1 > k_2$. In this case $R_{t_{\text{crit}}}$ for possible finger instability is shown by (11) if $1 - S_1 S_2$ and $S_1 + 1$ are of the same signs and by (12) if they are of opposite signs.

(i) $S_1 + 1 > 0$

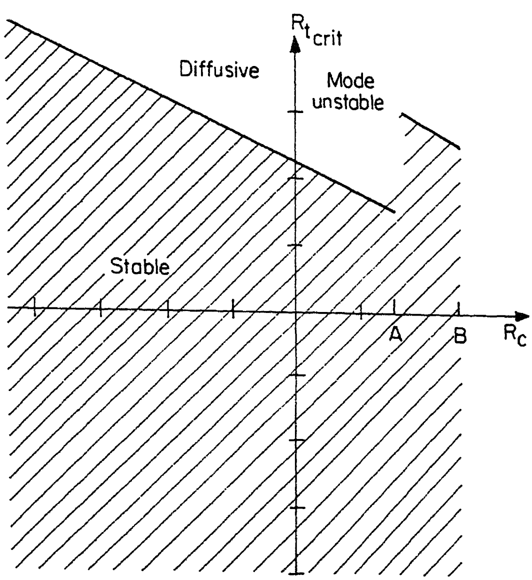

Figure 1. Stability diagram in the R_c–$R_{t_{\text{crit}}}$ plane showing the discontinuity of $R_{t_{\text{crit}}}$ for $S_1 + 1 = 0$ and $S_2 + 1 > 0$. $A = \pi^2(1 + k_1^{-1/2})^2$, $B = \pi^2(1 + k_2^{-1/2})^2$.

When $1 - S_1 S_2 > 0$, for the values of R_c for which (16) is satisfied for a given medium (i.e. k_1, k_2 known) diffusive instability sets in with $R_{t_{crit}}$, a_c given by (13) and for the values of R_c for which (16) is not satisfied, the inequality (16) with k_1 replaced by k_2 is to be checked for the possible existence of diffusive instability, i.e. for the values of R_c for which

$$R_c[(1 + S_2) - \tau(1 + S_1)] + \pi^2(1 + k_2^{-1/2})^2$$
$$[(1 + \tau)(1 + S_1) - (1 - S_1 S_2)] < 0 \tag{17}$$

a comparison of $R_{t_{crit}}$ given by (11) and (14) decides the mode of convection.

When $1 - S_1 S_2 < 0$, for the values of R_c for which (16) is satisfied, comparison of $R_{t_{crit}}$ given by (12) and (13) decides the mode of convection. For the values of R_c for which (16) is not satisfied, but (17) is satisfied, instability sets in through diffusive mode with $R_{t_{crit}}$ and a_c given by (14).

(ii) For values of S_1, S_2 which satisfy

$$S_1 + 1 > 0, \quad 1 - S_1 S_2 < 0 \text{ and } S_2 + 1 > 0 \tag{18}$$

it is seen from figure 2 that instability sets in even in a stable set-up (i.e. for some $R_c < 0$ and $R_t < 0$) which is caused by the presence of cross-effects which satisfy (16). It is also noted that, in this case, when the anisotropy parameter increases, the stable region in the R_c–$R_{t_{crit}}$ plane increases, implying thereby that an increase in the horizontal permeability in comparison with the vertical permeability inhibits convection, whereas for some other values of S_1, S_2 for which the diffusive instability exists, this is seen to aid convection.

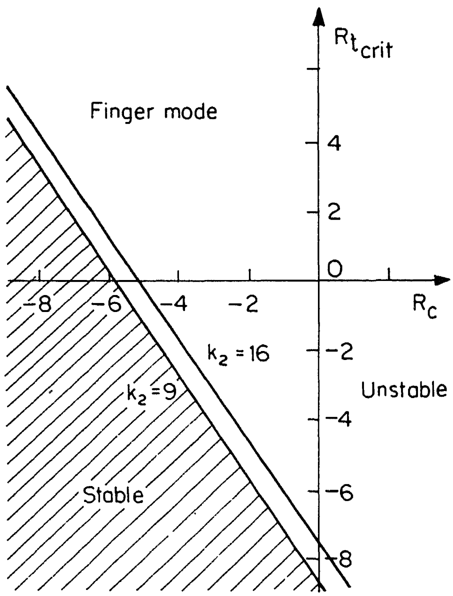

Figure 2. Diagram in the R_c–$R_{t_{crit}}$ plane showing the instability setting in even in a stable set-up for $S_1 + 1 > 0$, $S_2 + 1 > 0$, $1 - S_1 S_2 < 0$.

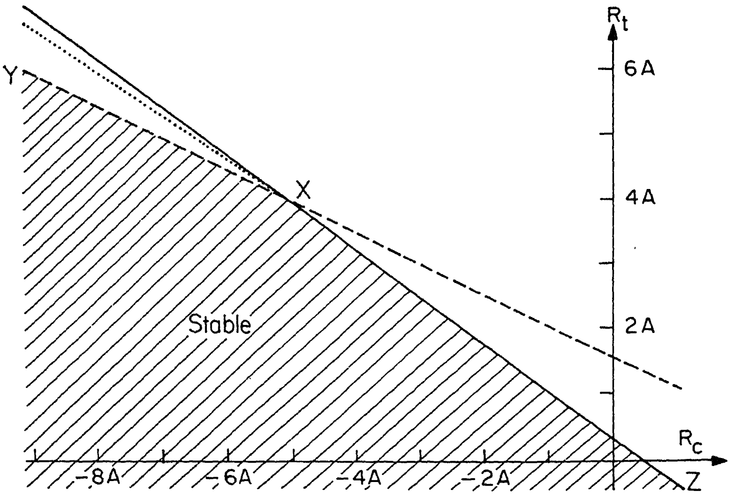

Figure 3. Stability diagram in the R_c–R_t plane. ——finger instability boundary ------diffusive instability boundary ..... transition from diffusive instability to finger instability. $A = \pi^2(1 + k_1^{-1/2})^2$.

(iii) Figure 3 shows the stability diagram in the R_c–R_t plane for a given wave number (which is here taken as $\pi k_1^{-1/4}$) and for S_1, S_2 which satisfy $S_1 + 1 > 0$, $S_2 + 1 > 0$, $1 - S_1S_2 > 0$. The critical Rayleigh numbers for finger mode and oscillatory mode are given by (11) and (13). For solute Rayleigh number R_c greater than a certain value R_c^* a direct (or finger) mode of disturbance is the most unstable one, while for $R_c < R_c^*$ an oscillatory (or diffusive) mode is the most unstable one where

$$R_c^* = \frac{(1 + S_1)(1 + \tau) - (1 - S_1S_2)}{\tau(1 + S_1) - (1 + S_2)} \pi^2(1 + k_1^{-1/2})^2.$$

The dotted curve in figure 3 represents the transition from diffusive mode to finger mode. This transition is given by equating to zero the discriminant of (10), assuming the wave number to retain its critical value, $\pi k_1^{-1/4}$. Similar results have been obtained by Tyvand [10] for thermohaline convection without cross-diffusion. ZX and XY represent the stability boundaries for finger and diffusive modes respectively.

(iv) $S_1 + 1 < 0$

It is seen, in this case, that finger instability sets in whether $1 - S_1S_2 > 0$ or $1 - S_1S_2 < 0$.

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References

[1] Chandrasekhar S, *Hydrodynamic and hydromagnetic stability* (Oxford: University Press) (1961)

[2] Cussler E L, *Multicomponent diffusion* (Elsevier Scientific Publishing Company: Oxford, New York Amsterdam) (1976)

[3] Hurle D T J and Jakeman E, Soret-driven thermosolutal convection, *J. Fluid Mech.* **47** (1971) 667–687

[4] Ingle S E and Horne F H, The Dufour effect, *J. Chem. Phys.* **59** (1973) 5882–5894

[5] Knobloch, Convection in binary fluids, *Phys. Fluids* **23** (1980) 1918–1920

[6] McDougall T J, Double-diffusive convection caused by coupled molecular diffusion, *J. Fluid. Mech.* **126** (1983) 379–397

[7] Neale G, Degrees of anisotropy for fluid flow and diffusion (electrical conduction) through anisotropic porous media, *Am. Inst. Chem. Eng. J.* **23** (1977) 56–62

[8] Nield D A, The thermohaline Rayleigh-Jeffreys problem, *J. Fluid Mech.* **29** (1967) 545–558

[9] Taslim M E and Narusawa U, Binary fluid convection and double-diffusive convection in a porous medium, *ASME J. Heat Transfer* **108** (1986) 221–224

[10] Tyvand P A, Thermohaline instability in anisotropic porous media, *Water Resour. Res.* **16** (1980) 325–330

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Canonical measures on the moduli spaces of compact Riemann surfaces

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Abstract. We study some explicit relations between the canonical line bundle and the Hodge bundle over moduli spaces for low genus. This leads to a natural measure on the moduli space of every genus which is related to the Siegel symplectic metric on Siegel upper half-space as well as to the Hodge metric on the Hodge bundle.

Keywords. Canonical measures; moduli spaces; compact Riemann surfaces; Hodge bundle; Siegel symplectic metric.

1. Introduction

As is very well-known today, Polyakov's (bosonic) string theory has produced a natural measure on the moduli spaces $\mathscr{M}_g$ $(g \geqslant 1)$. This measure has now been recognized as arising from pulling back the intrinsic Hodge metric of the Hodge line bundle E over $\mathscr{M}_g$ to the canonical bundle K over $\mathscr{M}_g$ via Mumford's (rather inexplicit) isomorphism $K \cong E^{13}$ for $g \geqslant 2$.* (See, for example, the exposition in Nelson [6].) In this article we first show that there are explicit and canonical isomorphisms $K \cong E^{(g+1)}$ for low genus $g = 1, 2, 3$; in the case of genus 3 the isomorphism $K \cong E^4$ is to be interpreted as holding over the Zariski open subset $\mathscr{M}_3^0 = \mathscr{M}_3 - \mathscr{h}_3$ of nonhyperelliptic surfaces. As a result, we can pull back the Hodge metric via our isomorphisms for these low genera to obtain a hermitian metric on K – and therefore another intrinsic volume form on $\mathscr{M}_1$, $\mathscr{M}_2$ and $\mathscr{M}_3 - \mathscr{h}_3$.

Because of the explicit nature of our isomorphisms we are able to actually exhibit this volume form in local (period matrix) coordinates on $\mathscr{M}_g$. To our surprise, it turned out that the volume form we are getting is nothing but the Riemannian volume form of Siegel's classical symplectic metric for $g = 1, 2, 3$. (For $g = 1$ this is the hyperbolic measure on $\mathscr{M}_1$.) In particular, our volume form assigns finite (and computable!) total volume to $\mathscr{M}_g$ for $g = 1, 2, 3$, whereas the Polyakov–Mumford measure of $\mathscr{M}_g$ is infinite for any g.

The above identification of our measure on $\mathscr{M}_g$ for low values of genus immediately allows us to extend our construction and define a corresponding measure on $\mathscr{M}_g$ for *every* genus by utilizing Siegel's symplectic metric on Siegel upper half-space $\mathscr{S}_g$. This is explained in §4. The results here may thus be interpreted as asserting that the

canonical line bundle K, (restricted to the Zariski open set of nonhyperelliptic points of $\mathcal{M}_g$ for $g \geqslant 3$), carries a natural hermitian metric which is closely related to *both* the Hodge metric of the Hodge bundle and the symplectic metric of Siegel space.

Our method of proof involves description of the Hodge bundle E and associated bundles over $\mathcal{M}_g$ for any g by explicitly finding the corresponding factors of automorphy for the action of the Torelli modular group on Torelli space $\mathcal{T}_g$. These explicit formulae (in §2 below) may hold some independent interest.

2. The moduli spaces and some vector bundles

The Teichmüller space, the Torelli space and the Riemann moduli space for compact Riemann surfaces of genus g ($\geqslant 1$) will be denoted respectively by T_g, $\mathcal{T}_g$ and $\mathcal{M}_g$. For $g \geqslant 3$ we will denote the $(2g-1)$ dimensional analytic subvariety of hyperelliptic Riemann surfaces in these three spaces by H_g, $\mathcal{H}_g$ and h_g respectively. (For $g=1$ and 2 all Riemann surfaces are hyperelliptic – but this will not affect us.) The Zariski open subsets of nonhyperelliptic points in these spaces, for $g \geqslant 3$, will be written T_g^0, $\mathcal{T}_g^0$ and $\mathcal{M}_g^0$. For the basic theory of these spaces, and for the material below, one may consult the book [5]. We set $T_g^0 = T_g$, $\mathcal{T}_g^0 = \mathcal{T}_g$ and $\mathcal{M}_g^0 = \mathcal{M}_g$ for $g = 1,2$.

Recall that the Teichmüller modular group Mod(g) and the Torelli modular group (which is identifiable with the symplectic group $\mathrm{Sp}(2g,\mathbb{Z})$) acts by biholomorphic automorphisms on the complex manifolds T_g and $\mathcal{T}_g$ respectively, producing $\mathcal{M}_g$ as the quotient normal complex space. These group actions keep the subsets T_g^0 and $\mathcal{T}_g^0$ invariant – giving $\mathcal{M}_g^0$ as quotient.

On a Riemann surface X of genus $g \geqslant 1$ we fix a standard homology basis $(\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g)$ for $H_1(X,\mathbb{Z})$, (characterized by having intersection matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$). The corresponding canonical dual basis $(\omega_1,\ldots,\omega_g)$ of holomorphic 1-forms on X is uniquely determined by the normalization requirement: $\int_{\alpha_j}\omega_i = \delta_{ij}$. The $g \times g$ matrix $\pi(X)$ of β-periods [i.e. $\pi_{ij}(X) = \int_{\beta_j}\omega_i$] is then called the *canonical period matrix* for the marked Riemann surface X. $\pi(X)$ is a symmetric matrix with positive definite imaginary part, and this gives us the usual *period mapping* $\pi:\mathcal{T}_g \to \mathcal{S}_g$ of Torelli space into the Siegel upper half-space. Recall that $\mathcal{S}_g$ is a hermitian symmetric domain of complex dimension $g(g+1)/2$, whereas $\mathcal{T}_g$ is of dimension 1 if $g=1$ and dimension $(3g-3)$ for $g>1$. It is known that π is a 2-to-1 holomorphic immersion of $\mathcal{T}_g^0$ into $\mathcal{S}_g$ for $g \geqslant 3$. For $g=1$ and 2 the period map is a holomorphic embedding of $\mathcal{T}_g$ onto its image in $\mathcal{S}_g$. Actually, for $g=1$, $\mathcal{S}_1 = \mathcal{T}_1 =$ upper half-plane U, and π can be regarded as the identity map. In genus 2, $\pi(\mathcal{T}_2)$ is "almost" all of $\mathcal{S}_2$. See [5] for more details.

We are interested in the relationship between certain important vector bundles over the moduli spaces. It is important to recall that the Teichmüller space T_g is a contractible domain of holomorphy, so that every holomorphic vector bundle over it is holomorphically globally trivial. Consequently, any bundle over $\mathcal{M}_g$ (or $\mathcal{T}_g$) can be described by a factor of automorphy for the corresponding modular group (or subgroup thereof) see Gunning [2]. The idea is to choose a global trivialization of the pull-back bundle over T_g and write down the action of the modular group on this pull-back.

One can construct the holomorphic vector bundles B_j, ($j \geqslant 1$), over T_g whose fiber

over $X \in T_g$ is the vector space of holomorphic j-forms on the Riemann surface X. One knows that for the bundle B_1 the canonical basis of 1-forms $(\omega_1(X), \dots, \omega_g(X))$ varies holomorphically with moduli (Bers [1]) – and so provides a *global holomorphic frame for B_1 over T_g*. Since, as we saw above, this global holomorphic frame is definable over the Torelli space $\mathscr{T}_g$, we see that the rank g bundle of 1-forms (we will still call it B_1) is holomorphically trivial over the Torelli space also. Let us describe the factor of automorphy for the action of the Torelli modular group $\Gamma = \mathrm{Sp}(2g, \mathbb{Z})$ on B_1 over $\mathscr{T}_g$ using this global trivialization.

Let $\gamma \in \mathrm{Sp}(2g, \mathbb{Z})$ be the symplectic matrix

$$\gamma = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \tag{1}$$

partitioned into $g \times g$ blocks. γ acts by changing the chosen standard homology basis $(\alpha, \beta) = (\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ into another standard homology basis $(\tilde{\alpha}, \tilde{\beta}) = (P\alpha + Q\beta, R\alpha + S\beta)$. If $\omega = (\omega_1(\tau), \dots, \omega_g(\tau))$ was the canonical basis of 1-forms at a point $\tau \in \mathscr{T}_g$ dual to the original homology basis (α, β), then we need to find the $GL(g, \mathbb{C})$ matrix that transforms ω to $\tilde{\omega} = (\tilde{\omega}_1, \dots, \tilde{\omega}_g)$, where the $\tilde{\omega}_j$ are the canonical dual 1-forms with respect to the new homology basis $(\tilde{\alpha}, \tilde{\beta})$. This requires that $\int_{\tilde{\alpha}_k} \tilde{\omega}_j = \delta_{jk}$; so a short calculation now shows that

$$\begin{bmatrix} \tilde{\omega}_1 \\ \vdots \\ \tilde{\omega}_g \end{bmatrix} = [A_\gamma] \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_g \end{bmatrix}, \tag{2}$$

where the sought-for matrix A_γ is:

$$A_\gamma = (P^t + \pi(\tau) Q^t)^{-1}. \tag{3}$$

(Here M^t denotes the transpose of a matrix M; $\pi(\tau) \in \mathscr{S}_g$, is the (canonical) period matrix of $\tau \in \mathscr{T}_g$, as defined before.) This A_γ is the factor of automorphy representing the 1-forms bundle over $\mathscr{M}_g$ as a Γ-quotient of the B_1 bundle over $\mathscr{T}_g$.

Now, the Hodge line bundle E is defined as the determinant bundle of B_1 (i.e. $E = \Lambda^g B_1$). *Therefore*

$$\xi(\gamma, \tau) = [\det(P + Q\pi(\tau))]^{-1}, \quad \gamma \in \mathrm{Sp}(2g, \mathbb{Z}), \quad \tau \in \mathscr{T}_g \tag{4}$$

is the factor of automorphy (with respect to the canonical trivialization) on Torelli space representing the Hodge bundle E over $\mathscr{M}_g$.

Remarks. (a) One can directly verify for formula (4) the cocycle condition

$$\xi(\gamma_1 \gamma_2, \tau) = \xi(\gamma_1, \gamma_2(\tau)) \cdot \xi(\gamma_2, \tau) \tag{5}$$

using the fact that the canonical period matrix at $\gamma(\tau)$ is

$$\pi(\gamma(\tau)) = (R + S\pi(\tau)) \cdot (P + Q\pi(\tau))^{-1}. \tag{6}$$

(b) To see whether a line bundle represented by a factor of automorphy $\xi(\gamma, \tau)$ is trivial, one has to find whether it has a nowhere vanishing global holomorphic section. Such a section exists if and only if there is a nowhere vanishing holomorphic function

φ on the covering space ($\mathscr{T}_g$ here) such that

$$\xi(\gamma, \tau) = \varphi(\gamma(\tau))/\varphi(\tau), \quad \text{for } \gamma \in \Gamma, \quad \tau \in \mathscr{T}_g. \tag{7}$$

In particular, we can think of $\operatorname{Pic}(\mathscr{M}_g)$ as equivalence classes of factors of automorphy for the Teichmüller modular group $\Gamma = \operatorname{Mod}(g)$ on T_g, where ξ_1 and ξ_2 are equivalent if $\xi_1 \cdot \xi_2^{-1}$ is trivial in the above sense. This remark will be invoked in the next section.

3. The isomorphism $K \cong E^{g+1}$ in low genus

The canonical bundle $K(M)$ of a complex space M is the determinant line bundle of the holomorphic cotangent bundle of M. It is a basic fact that the holomorphic cotangent bundle of the moduli spaces (T_g, $\mathscr{T}_g$ or $\mathscr{M}_g$) is identifiable, by Teichmüller's lemma (see [5]), as the bundle B_2.

Since the symmetric tensor product of two 1-forms on a Riemann surface X is a 2-form, we have a natural vector bundle map v of the second symmetric tensor power of B_1 to B_2 (say over Torelli space):

$$S^2(B_1) \xrightarrow{v} B_2. \tag{8}$$

By Max Noether's well-known theorem one sees that *the map v is a surjection over all of $\mathscr{T}_1$ and $\mathscr{T}_2$, and over $\mathscr{T}_g^0 = \mathscr{T}_g - \mathscr{H}_g$ for $g \geqslant 3$.* Note that, for $g \geqslant 2$, $S^2(B_1)$ is of rank $g(g+1)/2$ but B_2 is of rank $(3g-3)$. (These two numbers are equal precisely when $g = 2$ or 3.) The fiber above X of the kernel of the above bundle map v corresponds to linear dependence relations amongst the $g(g+1)/2$ quadratic differentials

$$\theta_{ij} = \omega_i \otimes \omega_j \quad \text{on } X. \tag{9}$$

These "Noether relations" are actually nothing but differential versions of relations amongst the π_{ij} which describe the Schottky locus (i.e. the image of $\mathscr{T}_g$ in $\mathscr{S}_g$).

We can now state

Theorem 1. *There are canonical analytic isomorphisms of line bundles:*

(i) $K \cong E^2$ *on* $\mathscr{M}_1$

(ii) $K \cong E^3$ *on* $\mathscr{M}_2$

(iii) $K \cong E^4$ *on* $\mathscr{M}_3 - \mathscr{h}_3 = \mathscr{M}_3^0$.

Proof. The global holomorphic frame $\omega = (\omega_1, \ldots, \omega_g)$ for B_1 over $\mathscr{T}_g$ gives rise (via v) to a global holomorphic frame $\theta_{ij} = \omega_i \otimes \omega_j$ ($(i,j) = 1, \ldots, g(g+1)/2$) for B_2 over $\mathscr{T}_g^0$ for $g = 1, 2, 3$. (For $g = 1, 2$ we get $\mathscr{T}_g^0 \equiv \mathscr{T}_g$.) Any element $\gamma \in \operatorname{Sp}(2g, \mathbb{Z})$ of the Torelli modular group acts on $(\omega_1, \ldots, \omega_g)$ by the matrix A_γ of formula (3). Consequently, the corresponding automorphism on the θ_{ij}-frame for B_2 is the second symmetric power $S^2(A_\gamma)$ of A_γ. The question therefore reduces to relating the determinant of $S^2(A)$ to the determinant of a linear automorphism A. Indeed, the following general algebra lemma shows that $\det(S^2(A)) = (\det(A))^{g+1}$, where A operates on a g-dimensional vector space. We are through. $\square$

Multilinear algebra lemma. *Let $A: V \to V$ be a linear automorphism of a g-dimensional vector space. Let $S^k(A): S^k(V) \to S^k(V)$ be the corresponding automorphism on the* k*th symmetric tensor powers of V. Then*

$$\Lambda^{\binom{g+k-1}{k}} S^k(A) = [\Lambda^g(A)]^{\binom{g+k-1}{k} \cdot \frac{k}{g}}. \tag{10}$$

Remarks. $S^k(V)$ has dimension $\binom{g+k-1}{k}$; so the left side of (10) is $\det(S^k(A))$. Also note that the product of the exponents on the two sides of (10) match! [It is quite instructive to write down the rather pretty (functorially determined) matrix for $S^k(A)$ from the matrix for a general A. The relation between $\det(A)$ and $\det(S^k(A))$ then appears rather remarkable.]

Proof of lemma. We can choose a basis $(e_1, \ldots, e_g)$ of V such that A has upper triangular matrix in this basis (assume first that V is a complex vector space). Then, in the corresponding induced basis of $S^k(V)$ one recognizes that $S^k(A)$ has again upper triangular matrix! Thus now $\det(S^k(A))$ is the product of just the diagonal entries in $S^k(A)$ – and these depend *only* on the diagonal entries of A. One therefore sees that $\det(S^k(A))$ must be some (universal) power of $\det(A)$. It is now easy to verify that the exponents are as in (10). (For example, it is now enough to just check the result for $A = \operatorname{diag}(d, d, \ldots, d)$. Or one can do some combinatorial counting of exponents here.)

Finally note that, once having proved this over complex vector spaces, we are through in general because (10) is nothing but a polynomial identity with integer coefficients, (this is a "principle of permanence of analytic relationships"!) □

COROLLARY 1.

Let E be the Hodge line bundle over $\mathscr{M}_g$.

(a) *The bundle E^{10} on $\mathscr{M}_2$ is globally analytically trivial.*

(b) *The bundle E^9 on $\mathscr{M}_3$ is the line bundle of the divisor given by some integer times the hyperelliptic divisor $\mathscr{h}_3$.*

Proof. These facts follow by combining our Theorem 1 with Mumford's [3] theorem that $K \cong E^{13}$ on $\mathscr{M}_g$ for $g \geqslant 2$. It is to be remembered that $\mathscr{h}_3$ is a codimension 1 connected analytic subvariety in $\mathscr{M}_3$. □

Remarks. Mumford in [4] finds that $\operatorname{Pic}(\mathscr{M}_2) = \mathbb{Z}/10\mathbb{Z}$ by looking at the geometry of the Teichmüller modular group. Our present corollary (a) confirms this from quite a different angle. Since we know explicitly from formula (4) the factor of automorphy for E, hence for E^{10}, it may be interesting to prove directly the triviality of E^{10} on $\mathscr{M}_2$ by finding the nowhere vanishing holomorphic φ on $\pi(\mathscr{T}_2)$ (from theta-nulls) which exhibit that $\xi^{10} = \varphi(\gamma(\tau))/\varphi(\tau)$. (See (7) of §2.)*.

***Note added in proof**

(G Gonzales-Dies has answered this (thesis, King's College, London) by showing that $\varphi(\tau) = \prod_{\varepsilon\,\text{even}} \theta^2[\varepsilon](0, \pi(\tau))$—i.e., the product of the squares of the 10 even theta-nulls—gives a nowhere vanishing section of E^{10} over $\mathscr{M}_2$ (Private communication.)

A question. To describe K on $\mathscr{M}_g$, for $g \geqslant 4$, by a factor of automorphy for $\mathrm{Sp}(2g, \mathbb{Z})$ we need to know whether the canonical bundle (or better still B_2) over Torelli space is trivial. *We have seen above that* K *(and* B_2*) over* $\mathscr{T}_g^0$ *is trivial for* $g = 1, 2, 3$. Does this phenomenon persists in higher genus?

4. A canonical measure on $\mathscr{M}_g$ for all g

The Hodge bundle E, and more so B_1 itself, carries the completely intrinsic *Hodge hermitian metric* on its fibers. This is given by pairing any two holomorphic 1-forms φ and ψ on a Riemann surface X as follows:

$$(\varphi, \psi) = i \iint_X \varphi \wedge \bar{\psi}. \tag{11}$$

Theorem 2. *The Hodge hermitian metric on* E^{g+1} $(g = 1, 2, 3)$ *pulled back by the isomorphisms of Theorem 1 produces a hermitian metric on* K *(over* $\mathscr{M}_1$, $\mathscr{M}_2$ *and* $\mathscr{M}_3^0$*). This gives a volume form on* $\mathscr{M}_g$ $(g = 1, 2, 3)$ *which can be explicitly written down as*

$$\mathrm{d(vol)} = \frac{\bigwedge_{(i,j)=1}^{g(g+1)/2} (\mathrm{d}\pi_{ij} \wedge \mathrm{d}\bar{\pi}_{ij})}{[\det(\mathrm{Im}\, \pi_{ij})]^{g+1}} \tag{12}$$

in period-matrix coordinates π_{ij} *for* $\mathscr{T}_g \subset \mathscr{S}_g$. *(Thus,* d(vol) *is a volume form on Siegel upper half-space* $\mathscr{S}_g$ *which is invariant under the Torelli modular group.)*

Remarks. (i) For $g = 1$. d(vol) is just the hyperbolic (Poincare volume) measure: $(\mathrm{d}\tau \wedge \mathrm{d}\bar{\tau})/(\mathrm{Im}\,\tau)^2$ on upper half τ-plane.

(ii) For any g the formula (12) as a volume form on $\mathscr{S}_g$ is nothing but (apart from a constant factor) the Riemannian volume form obtained from Siegel's symplectic hermitian metric on $\mathscr{S}_g$:

$$\mathrm{d}s^2 = \mathrm{trace}\,[(\mathrm{Im}\,\pi_{ij})^{-1}(\mathrm{d}\pi_{ij})(\mathrm{Im}\,\pi_{ij})^{-1}(\mathrm{d}\bar{\pi}_{ij})]. \tag{13}$$

This metric, and hence its volume element (12), is invariant under the *full* real symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ (and not just $\mathrm{Sp}(2g, \mathbb{Z})$). The volume element for (13) can be seen in Siegel [8], p. 130. It is $1/2^g$ times the volume element from (12).

Proof of Theorem. The canonical frame $(\omega_1, \ldots, \omega_g)$ for B_1, and the induced frame $\theta_{ij} = \omega_i \otimes \omega_j$ for B_2, give holomorphic trivializations of B_1 and B_2 over $\mathscr{T}_g^0$ for these low genera. Pulling back the Hodge norm from E^{g+1} to K we see that

$$\left\| \bigwedge_1^{g(g+1)/2} \theta_{ij} \right\|^2 = [\det(\omega_i, \omega_j)]^{g+1}.$$

The induced modular-invariant volume form on $\mathscr{T}_g^0$ is therefore

$$\mathrm{d(vol)} = \frac{\bigwedge_1^{g(g+1)/2} (\theta_{ij} \wedge \bar{\theta}_{ij})}{[\det(\omega_i, \omega_j)]^{g+1}}. \tag{14}$$

By Riemann's bilinear relations one sees that the denominator is the same as the denominator in (12). Now, the quadratic differentials θ_{ij} have to be interpreted as cotangents to Torelli space, using Teichmüller's lemma. A famous variational formula of Rauch does precisely that (for $g \geqslant 2$) and says

$$\omega_i \otimes \omega_j = \mathrm{d}\pi_{ij}. \tag{15}$$

See [5], pp. 260–263 for a proof. Therefore, (15) and (14) prove (12) (for $g \geqslant 2$ at least).

In genus 1 it is directly possible to identify $\omega_1^2 = \theta_{11}$ as the cotangent vector $\mathrm{d}\tau$, where the upper-half τ-plane U is the Teichmüller and the Torelli space $T_1 = \mathcal{T}_1 = U$.

The normalized ω_1 on a torus $X_\tau = \mathbb{C}/L(1, \tau)$ is clearly $\mathrm{d}z$ (here the α-loop is the projection of the segment $[0, 1]$ and the β-loop is the projection of the segment $[0, \tau]$). Now one writes down the affine (Teichmüller mapping) f of X_τ onto X_σ, and calculates its Beltrami coefficient $\mu = \bar{\partial} f / \partial f$ on X_τ. The formula for μ is

$$\mu = \left(\frac{\tau - \sigma}{\sigma - \bar{\tau}} \right) \frac{\mathrm{d}\bar{z}}{\mathrm{d}z} \quad \text{on } X_\tau, \text{ for any } \tau, \sigma \in U. \tag{16}$$

We compute the Teichmüller pairing $\iint \mu \theta_{11}$ of μ and $\theta_{11} = \mathrm{d}z^2$ to get

$$\begin{aligned} (\mu, \theta_{11}) &= \int_{X_\tau}\!\!\int \left(\frac{\tau - \sigma}{\sigma - \bar{\tau}} \right) \cdot \mathrm{d}z \wedge \mathrm{d}\bar{z} \\ &= -2i \left(\frac{\tau - \sigma}{\sigma - \bar{\tau}} \right) \cdot (\text{area of period parallelogram for } X_\tau). \end{aligned} \tag{17}$$

Setting now $\sigma = \tau + \varepsilon\, \mathrm{d}\tau$, and letting $\varepsilon \to 0$, we get

$$(\mu, \theta_{11}) = \varepsilon\, \mathrm{d}\tau + O(\varepsilon^2). \tag{18}$$

This shows that θ_{11} corresponds to $\mathrm{d}\tau$, as required. □

From Theorem 2 and Remark (ii) we now see the correct generalization of the measure we are getting in low genera. *Take the Riemannian metric induced on* $\mathcal{T}_g^0 = \mathcal{T}_g - \mathcal{H}_g$ *from the symplectic metric* (13) *on* $\mathcal{S}_g$, ($\mathcal{T}_g^0$ *is immersed in* $\mathcal{S}_g$ *by the period mapping* π), *and then take the corresponding volume form on* $\mathcal{T}_g^0$ (*which is certainly modular-invariant*). *We thus get a volume form on* $\mathcal{M}_g^0$ *for every g.*

Siegel [9], p. 6, computed the (finite) volume of $\mathcal{S}_g / \mathrm{Sp}(2g, \mathbb{Z})$ with the volume element of (13). Thus our measures give *finite* total mass to $\mathcal{M}_g$ at least for $g = 1, 2, 3$. We do not know at present whether the volume of $\mathcal{M}_g$ will be finite for $g \geqslant 4$ because the Schottky problem ($\mathcal{T}_g \subset \mathcal{S}_g$) complicates matters thoroughly.

Remark. Note that no simple expression like (12) was possible for the Polyakov measure on $\mathcal{M}_g$ because the Mumford isomorphism is inexplicit, so one does not know which wedge of holomorphic quadratic differentials should correspond to $(\omega_1 \wedge \cdots \wedge \omega_g)^{13}$.

5. Another related canonical metric on T_g

The Riemannian metric on T_g^0 induced from Siegel's metric (13) on Siegel's upper half-space came up, as we saw, very naturally in the calculations of this paper. This

metric had previously been given consideration by Royden [7]. This metric on moduli has a close connection to another completely natural hermitian metric on T_g induced via the metric on the Riemann surface X obtained from the embedding of X in its flat canonically polarized Jacobi variety.

We conclude with some explanations and open questions regarding this last metric mentioned above. Notice that given any canonical choice of a (conformal) Riemannian metric on a (varying) Riemann surface X, there is a naturally induced metric on the Teichmüller space $T(X)$. Indeed, to define a hermitian (co-)metric on $T(X)$ one needs to assign a hermitian inner product on the space of holomorphic quadratic differentials on X (since these comprise the cotangent vectors to $T(X)$). Such a pairing is always definable by the formula

$$\langle \varphi, \psi \rangle = \int_X\!\!\int \varphi \bar{\psi} H^{-1} \tag{19}$$

where $ds^2 = H(z)|dz|^2$ is the chosen conformal metric on X. For example, when ds^2 is chosen as the hyperbolic metric, we obtain the Weil–Peterson metric on $T(X)$ by this construction (see, for example, [5], p. 404.).

Our idea is to use the metric on the compact Riemann surface obtained from the Abel–Jacobi embedding $X \hookrightarrow J(X)$, where $J(X)$ is equipped with its canonical flat metric. In fact,

$$J(X) = A(X)^*/H_1(X, \mathbb{Z}), \tag{20}$$

where $A(X)$ is the g-dimensional vector space of holomorphic 1-forms on X. The Hodge inner product (11) on $A(X)$ gives a dual inner product to the dual vector space, and this makes $J(X)$ a flat torus in the canonical way. The pull-back metric on X is a conformal (Kähler) metric which assigns total area g to X, and has non-positive curvature everywhere on X. These assertions are very easy to verify. Indeed, note that the curvature inequality follows from general principles because X is embedded as a minimal surface in $J(X)$; (any Kähler submanifold of a Kähler manifold is a minimal variety). So the two principal curvatures are equal and opposite (mean curvature must vanish) – hence the Gaussian curvature is non-positive on X, as stated.

A computation shows that the local expression for this Jacobian-induced metric on X is $ds^2 = H(z)|dz|^2$ where

$$H(z) = \sum_{j,k=1}^{g} \omega_j(z) \lambda_{jk} \overline{\omega_{k(z)}}. \tag{21}$$

Here $(\omega_1, \dots, \omega_g)$ is the standard Riemann normalized basis for $A(X)$ (see §2), and the λ-matrix is the inverse of the positive-definite $(\operatorname{Im} \pi_{ij})$ matrix. $H(z)$ can also be expressed as $k(z,z)$ where

$$k(z, \zeta) = \sum_{j=1}^{g} u_j(z) \overline{u_j(\zeta)} \tag{22}$$

is the reproducing kernel for $A(X)$ with respect to the Hodge pairing (11). Here $(u_1, \dots, u_g)$ is any orthonormal basis for $A(X)$ with respect to (11). (Note: $k(z, \zeta)$ is a "bi-Abelian differential" on X and $H(z) = k(z,z)$ is an area form.)

Using this H in formula (19) we thus obtain a natural "Jacobian-induced" hermitian metric on T_g. One checks that this metric is modular invariant on T_g because of the naturality of the construction.

The Jacobian-induced metric on T_g and the Siegel-space-induced metric are comparable via the expressions shown for each. Therefore, in order to study questions like finite volume for $\mathscr{M}_g$ in the Siegel metric, and also for many other reasons as well as for its own sake, it would be interesting to find out whether this Jacobian-induced metric on T_g is (i) complete or not; (ii) has negative curvature or not; (iii) gives finite volume for $\mathscr{M}_g$ or not. We hope to report on these matters in the future.

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References

[1] Bers L, Holomorphic differentials as functions of moduli, *Bull. Am. Math. Soc.* **67** (1961) 206–210
[2] Gunning R, *Riemann surfaces and generalized theta functions* (New York: Springer-Verlag) (1976)
[3] Mumford D, Stability of projective varieties, *L. Ens. Math.* **24** (1977) 39–110
[4] Mumford D, Abelian quotients of the Teichmüller modular group, *J. d'Anal. Math.* **18** (1967) 227–244
[5] Nag S, *The complex analytic theory of Teichmüller spaces* (New York: John Wiley Interscience) (1988)
[6] Nelson P, Lectures on strings and moduli space, *Physics Reports* (1987)
[7] Royden H, Invariant metrics on Teichmüller space, in *Contributions to analysis* (eds) L Ahlfors *et al* (New York: Academic Press) (1974)
[8] Siegel C L, *Topics in complex function theory* (New York: John Wiley Interscience) Vol. 33 (1973)
[9] Siegel C L, *Symplectic geometry* (New York: Academic Press) (1964)

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Dynamics on Ahlfors quasi-circles

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Abstract. The celebrated theory of Denjoy introduced a topological invariant distinguishing C^1 and C^2 diffeomorphisms of the circle. A C^2 diffeomorphism of the circle cannot have an infinite minimal set other than the circle itself. However, this is possible for C^1 diffeomorphisms. In dimension two there is a related invariant distinguishing C^2 and C^3 diffeomorphisms.

Theorem. Let Q be a quasi-circle contained in a surface. If Γ is an infinite minimal isometry set in Q for a C^3 diffeomorphism, then Γ equals Q. There exists a C^2 diffeomorphism of the annulus with a minimal Cantor set contained in a quasi-circle.

Keywords. Quasi-circle; minimal set; rotation number; Cantor sets; Denjoy counterexample.

1. Introduction

Poincaré and Birkhoff proved that a measure preserving homeomorphism of the two-dimensional annulus which twists the two boundary components in opposite directions must have fixed points in the interior of A. What is known as KAM theory emerged from this and is currently being developed and refined. (See [7], [8] and [9], for example.) The theory produces global topological and dynamical conclusions from local assumptions—C^3 differentiability and infinitesimal 'twisting'. The C^3 hypothesis is sharp; there exist counterexamples in the $C^2 + \delta$ category. (See [6].) In this paper we consider a problem with some rudimentary resemblance to twist theory.

Let f be a C^r diffeomorphism of the two-dimensional annulus $A = S^1 \times [-1, 1]$ to itself which is repelling at one boundary component $A^+ = S^1 \times \{1\}$ and attracting at the other $A^- = S^1 \times \{-1\}$. Suppose f has no periodic points in $\mathrm{int}(A_0)$. If one orbit 'gets across', must they all? That is, if the α-limit set of x_0 is in A^+ and its ω-limit set is in A^- for some $X_0 \in A$, then is this true for every $X \in A$? There is growing evidence that the answer is in the affirmative for $r = 3$. We pose an equivalent version of this question.

2. The north pole, south pole conjecture

Suppose $f: S^2 \to S^2$ is a C^3 diffeomorphism with the north pole N a repeller, the south pole S an attractor and no other periodic points. If one orbit is asymptotic to both

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S and N then f is dynamically equivalent to the standard north pole, south pole diffeomorphism.

Formally, the conclusion means that f is the time one map of the gradient flow on S^2.

A C^r diffeomorphism $f:M \to M$ is of class $C^{r+\delta}$ if the rth derivative satisfies a δ-Hölder condition. That is, there exists $C > 0$ such that $\| D^r f_x - D^r f_y \| < C \| x - y \|^\delta$ for $x, y \in M$.

There exist counter-examples to the NP-SP conjecture if f is $C^{2+\delta}$. (See [5], [4] and [3].) We show in this paper that C^3 is a natural bound to these examples. Hence the NP-SP conjecture concerns the topological-dynamical invariants that might distinguish $C^{2+\delta}$ and C^3.

We find the coordinates of the annulus more convenient to work with and put this spherical formulation aside.

A non-empty, closed, invariant set Γ of a homeomorphism f is said to be *minimal* if it is closed and contains no smaller non-empty, closed, invariant sets.

If there is a counter-example to the NP-SP conjecture then there exists a C^r diffeomorphism f of the annulus without periodic points which has one orbit asymptotic to both boundary components and one orbit whose closure Γ stays bounded away from ∂A. Furthermore, Γ may be taken to be a minimal set. In [4] it is shown that the existence of such a diffeomorphism implies the existence of a C^r Seifert counter-example. That is, there exists a C^r vector field on the three-sphere S^3 with neither zeroes nor closed integral curves. Hence the NP-SP conjecture is 'contained' in the Seifert conjecture.

A recent theorem of John Franks is useful in analyzing the dynamics of f.

Theorem (Franks). *Let $f:A \to A$ be a homeomorphism of the open annulus A and $x \in A$. Let g be a lift of f to the universal cover $\mathbb{R}\times[-1,1]$ of A and y a lift of x. Let y_n denote the first component of $g^n(y)$. If there exists a rational number p/q with*

$$\underline{\lim}\frac{y_n}{n} \leqslant \frac{p}{q} \leqslant \overline{\lim}\frac{y_n}{n}$$

then there exists a point $z \in A$ with $f^q(z) = z$.

By the theorem of Franks, Γ must be an infinite, perfect minimal set which has irrational rotation number—the cyclic order is preserved by f. Certainly Γ could not be a circle, otherwise no orbit would get across it; however, it is not known if Γ must be a Cantor set.

3. The Denjoy Cantor sets

The reader might be reminded of the Denjoy's theory where the critical degree of differentiability is 2 and the dimension is 1. Denjoy [1] found that the degree of differentiability of a circle diffeomorphism f influences its topological type. If f is C^2 and has no periodic points then f has simple dynamics—it is topologically conjugate to a rotation through an irrational angle. This is not the case in the $C^{1+\delta}$ category.

Suppose $\Gamma \subset S^1$ is a Cantor set. If there exists a homeomorphism $f:S^1 \to S^1$ for which Γ is minimal, then the pair (f, Γ) is a *Denjoy Cantor set.* Denjoy Cantor sets provide the key ingredient to classifying homeomorphisms of the circle. Poincaré defined the *rotation numbers* and showed that all homeomorphisms of the circle have

them. Furthermore, any homeomorphism of the circle with irrational rotation number α is either topologically conjugate to a rigid rotation through α or has a minimal Cantor set. Denjoy proved that these examples can all exist as C^1 diffeomorphisms but not C^2 (actually C^{1+bv} is impossible). We call these examples Denjoy Cantor sets. More generally, a homeomorphism $g:\Gamma' \to \Gamma'$, Γ' contained in an n-manifold M is also called a Denjoy Cantor set if the pair (g, Γ') is topologically conjugate to a Denjoy Cantor set (f, Γ) in S^1. That is, there exists an embedding $h:\Gamma \to M$ such that $h(\Gamma) = \Gamma'$ and $h \cdot f = g \cdot h$.

It is not completely understood under what circumstances Denjoy Cantor sets can exist. Hall [2] showed that it is possible to have a Denjoy Cantor set in a C^∞ annular diffeomorphism. However, it is attracting, and so no orbit is asymptotic to both boundary components. Do there exist C^3 diffeomorphisms f of A with no periodic points, a Denjoy Cantor set (f, Γ) and one orbit asymptotic to both boundary components? We consider some possibilities.

There are two features of Γ for us to study—its structure as a subset of A and the properties of the first derivative of f at Γ, the *distortion* of f at Γ.

Using the methods of Denjoy, one can rule out any Denjoy Cantor set Γ contained in a smooth Jordan curve as long as f is C^2. It is not possible for Γ to have totally arbitrary topological structure since minimality implies homogeneity. A natural question arises—how wild can Γ be?

In Denjoy's theory, the simplest C^1 examples have first derivative, the identity at the minimal Cantor set. It is quite easy to show there are no C^2 diffeomorphisms of the circle with this condition at the Cantor set: Let L_n denote the intervals complementary to Γ, indexed so that $f(L_n) = L_{n+1}$. Let $a_n = |L_n|$. Since f is C^1, we can apply the mean value theorem and continuity of the first derivative to conclude that $a_n/a_{n+1} \to 1$ as $n \to \infty$. If f were C^2, we could apply the mean value theorem to f' and use continuity of the second derivative to conclude that

$$\frac{1 - \dfrac{a_n}{a_{n+1}}}{a_n} \to 0 \quad \text{as } n \to \infty.$$

It follows that $\sum a_n = \infty$, contradicting the finite arc length of S^1.

The same proof extends to $\mathbb{R}^2$ and shows there are no C^3 annular diffeomorphisms with the first derivative the identity and the second derivative the 0 bilinear transformation at a square rectifiable Denjoy Cantor set.

4. Isometry sets

In this paper we consider a large class of 'simplest' examples. We assume that the first derivative of f at each point of Γ is an isometry for some Riemannian metric on A. The isometry may vary from point to point. We call Γ an *isometry* set. Note that this isometry condition on Γ, even for the usual metric on A, is weaker than the identity hypothesis. The annulus may be replaced by any Riemannian manifold M.

We need a little more background before we can state the main results.

Smooth curves Q are *n-rectifiable* for all $n \geq 1$. That is, there exists a constant $L > 0$ such that $\sum |x_i - x_{i+1}|^n < L$ for all partitions $x_1 < x_2 < \cdots < x_i < x_{i+1} < \cdots$ of Q. A curve is *square rectifiable* if it is 2-rectifiable. One can similarly define the notion of

n-rectifiable for subsets of curves or for any set on which there is a well-defined order. In particular, we may consider whether Denjoy Cantor sets are n-rectifiable. Topological curves and Cantor sets may be so wild that they fail to be n-rectifiable, for any n. However, additional restrictions guarantee that curves be square-rectifiable.

A curve Q is called a *quasi-curve* if there exists $K > 0$ such that if x, $y \in Q$, the arc connecting x and y is contained in a disc of radius $Kd(x, y)$. A *quasi-circle* is a Jordan curve which is the union of quasi-arcs. We prove

PROPOSITION

Quasi-arcs are square rectifiable.

Theorem. *Let f be a C^3 diffeomorphism of a compact Riemannian n-manifold M and $\Gamma \subset M$ a minimal isometry set. If Γ is a Denjoy Cantor set then Γ is not square rectifiable.*

The Proposition and Theorem imply the following

COROLLARY

Let f be a C^3 diffeomorphism of the annulus A and $Q \subset A$ a quasi-circle. If $\Gamma \subset Q$ is an infinite, minimal isometry set then $\Gamma = Q$.

Proof. Since Γ is an infinite minimal set, the rotation number of $f|_\Gamma$ is irrational. Then Γ can only be a Cantor set or all of Q.

These results depend on a general estimate for the asymptotic behaviour of pairs of orbits of isometry minimal sets. This is an example of an estimate of 'non-linear' distortion.

Theorem. *Let E be an isometry minimal set of a C^3 mapping f of a compact Riemannian 2-manifold M. If $y \in M$ and $z \in E$ then*

$$\sum_{n=1}^{\infty} \mathrm{d}(y_n, x_n)^2 = \infty.$$

The proof of Denjoy's result depends on the divergence of the Poincaré series for C^2 maps f

$$\sum_{n=1}^{\infty} |Df_x^n|^1.$$

(See Sullivan [10], for example). In practice Df_x^n is sometimes replaced by $\mathrm{d}(x_n, y_n)$ for s in the minimal set and y arbitrary in the manifold where d is the Riemannian metric on M. The exponent is related to the degree of differentiability of f. If f is C^r, it is natural to estimate the general dynamic sum

$$\sum_{n=1}^{\infty} \mathrm{d}(y_n, X_n)^{r-1}$$

ven on higher dimensional manifolds. In this paper we restrict ourselves to emannian two-manifolds and $r = 3$, although generalizations to higher dimensions d degrees of differentiability are possible.

5. Isometry and the second derivative

A linear transformation from one normed space to another, say $T{:}E_1 - E_2$, is an *isometry* if T is a bijection and

$$\| Tx \|_{E_2} = \| x \|_{E_1} \text{ for all } x \in E_1.$$

The quantity

$$\rho(T) = \max\left[\sup_{\|x\|_{E_1}=1} \| Tx \|_{E_2},\ \sup_{\|x\|_{E_1}=1} 1/\| Tx \|_{E_2}\right]$$

measures how non-isometric is T. Then $\rho(T) \geqslant 1$; T is an isometry if and only if $\rho(T) = 1$ and T is a bijection.

Lemma 5.1. Let f be a C^2 diffeomorphism of a Riemannian m-manifold M and $p \in M$, fixed. Suppose there exists $p_n \to p$ such that $T_{p_n} f$ is an isometry respecting the given Riemann structure g of M. Let $\bar{f}$ be the lift of f from M to $T_pM \approx R^m$ at p.

$$\begin{array}{ccc} T_pM & \xrightarrow{\bar{f}} & T_{fp}M \\ \Big\downarrow{\scriptstyle \exp_p} & & \Big\downarrow{\scriptstyle \exp f_p} \\ M & \xrightarrow{f} & M \end{array} \qquad \bar{p}_n = \exp_p^{-1}(p_n)$$

Let $\bar{p}_n = \exp_p^{-1}(p_n)$, then the sequence of linear maps

$$(D\bar{f})\bar{p}_n{:}\quad T_pM - T_{fp}M$$

is non-isometric only to the extent:

$$|1 - \rho(D\bar{f}\bar{p}_n)| = O(\| \overline{p_n} \|^2)$$

Remarks. The norms on T_pM and $T_{fp}M$ are $g(p)$ and $g(fp)$ respectively. The length $\| \overline{p_n} \|$ is also calculated with respect to $g(p)$, although we could replace $\| \overline{p_n} \|$ with $d(p_n, p)$ since

$$\frac{\| \overline{p_n} \|}{d(p_n, p)} \to 1 \quad \text{as } p_n \to p.$$

Proof. To calculate $\rho((D\bar{f})\overline{p_n})$ one considers the pulled-up Riemann structure on T_pM and $T_{fp}M$, namely

$$\bar{g}_p{:}\, \bar{g}_p(w_p; u, v) = \langle T_{w_p} \exp_p(u), T_{w_p} \exp_p(v) \rangle_{\exp_p(w_p)}$$

for all $w_p \in T_pM$ near O_p and

for all $u, v \in T_{w_p}(T_pM) \approx T_pM$

$$\bar{g}_{fp}{:}\, \bar{g}_{fp}(w_{fp}; u, v) = \langle T_{w_{fp}} \exp_{fp}(u), T_{w_{fp}} \exp_{fp}(v) \rangle_{\exp_{fp}(w_{fp})}.$$

The map $D\bar{f}$ at the point $\overline{p_n} = \exp_p^{-1}(p_n)$ is an isometry from the tangent space to

T_pM at $\overline{p_n}$, equipped with the metric $\bar{g}_p(\overline{p_n};*)$ to the tangent space to $T_{fp}M$ at $\overline{f(p_n)}$ equipped with the metric $\bar{g}_{fp}(\overline{fp_n};*)$.

Let $e^1,\ldots,e^m$ be an orthonormal basis for T_pM.

The map $T_pf:T_pM - T_{fp}M$ is an isometry (being the limit of isometries) so $T_pf(e^1),\ldots,T_pf(e^m)$ is an orthonormal basis at $T_{fp}M$. Both these bases give rise to $\bar{g}_{ij}$-expressions for the metrics $\bar{g}_p$ and $\bar{g}_{fp}$ on T_pM and $T_{fp}M$. Besides,

$$\bar{g}_{ij}p(w_p) = \bar{g}_p(w_p; e^i, e^j) = \delta_{ij} + O(\|w_p\|^2)$$

$$\bar{g}_{ij}f_p(w_{fp}) = \bar{g}_{fp}(w_{fp}; T_pf(e^i), T_pf(e^j)) = \delta_{ij} + O(\|w_{fp}\|^2).$$

Thus

$$\langle (D\bar{f})\bar{p}_n(u), \quad (D\bar{f})\bar{p}_n(u)\rangle_{g(fp)}$$

$$= \sum (i\text{-th component of } v)^2 \text{ where } v = (D\bar{f})\overline{p_n}(u)$$

$$= \sum \delta_{ij}(i\text{-th component of } v)(j\text{-th component of } v)$$

$$= \sum \bar{g}_{ijfp}(\overline{fp_n})(i\text{-th component of } v)(j\text{-th component of } v)$$

$$+ \sum (\delta_{ij} - \bar{g}_{ij}f_p)(\overline{fp_n})(i\text{-th component of } v)(j\text{-th component of } v)$$

$$= \langle (D\bar{f})_{\overline{p_n}}(u), \quad (D\bar{f})_{\overline{p_n}}(u)\rangle_{\bar{g}_{fp}}(\overline{fp_n})$$

$$+ \sum (\delta_{ij} - \bar{g}_{ijfp})(\overline{fp_n})(i\text{-th component of } v)(j\text{-th component of } v)$$

$$= \langle u, u\rangle_{\bar{g}_p(\overline{p_n})} + O(\|\overline{fp_n}\|^2)\cdot \|D\bar{f}_{\overline{pn}}(u)\|^2$$

$$= \|u\|^2 + O(\|\overline{p_n}\|^2)\cdot \text{constant } \|u\|^2.$$

Since f is a diffeomorphism and T_pf is an isometry we have

$$\|\overline{p_n}\|/\|\overline{fp_n}\| \to 1 \quad \text{as } n \to \infty.$$

Hence $\|D\bar{f}_{\overline{p_n}}\| = (1 + O(\|\overline{p_n}\|^2)^{1/2} = 1 + O(\|\overline{p_n}\|^2)$. q.e.d.

Let x_1 be a sequence of points in $\mathbb{R}^n$. Suppose there exists a finite set of unit vectors $v^1, v^2,\ldots,v^m$ in $\mathbb{R}^n$ which is the limit set of $\{x_i/\|x_i\|\}$. If $\|x_i\| \to 0$, we say that the sequence x_i converges to 0 from m directions $v^1, v^2,\ldots,v^m$.

Lemma 5.2. Let g be a C^2 diffeomorphism of R^n. Suppose there exist points $x \in R^n$ converging to 0 from m directions $v^1,\ldots,v^m$ such that $|1-\rho(Dg_x)| = 0(\|x\|^2)$. Then $D^2g_0(v,w)\cdot Dg_0(u) = 0$ for $u, v, w \in sp(v^1,\ldots,v^m)$. If $\mathrm{sp}\,(v^1,\ldots,v^m) \cong R^n$ then $D^2g_0(v,w) = 0$.

The author is grateful to M Shub and C Robinson for the following proof.

Proof. Observe that Dg_0 is an isometry since $\rho(Dg_0) = 1$.

The set of linear transformations $\{Dg_0^{-1}Dg_x\}$ is tangent to the orthogonal metrices at the identity (where $x = 0$). The antisymmetric metrices form the tangent space to the orthogonal matrices based at the identity. Hence, if $A = Dg_0^{-1}D^2g_0$, then $A(v) = A_v$

is antisymmetric for $v \in \mathrm{sp}(v^1,\ldots,v^m\}$. Thus $A_v(w_1)\cdot w_2 = -w_1 \cdot A_v(w_2)$ for all vectors w_1 and w_2.

Let $v, w \in \mathrm{sp}\{v^1,\ldots,v^m\}$. Then $A_v(v)\cdot w = -v\cdot A_v(w) = -v\cdot A_w(v) = A_w(v)\cdot v = A_v(w)\cdot v = -w\cdot A_v(v) = 0$. Write $A_v(v) = x + y \in \mathrm{sp}\{v^1,\ldots,v^m\} \times R^l$ where R^l is the orthogonal complement to $\mathrm{sp}\{v^1,\ldots,v^m\}$. Then $0 = A_v(v)\cdot x = x\cdot x + y\cdot x$. Since $y\cdot x = 0$ we have $x\cdot x = 0$. Thus $x = 0$ and $A_v(v) \in R^l$. Since $A_{v+w}(v+w) = A_v(v) + 2A_v(w) + A_w(w) \in R^l$, then $A_v(w) \in R^l$. Hence $u\cdot A_v(w) = 0$ for all $u, v, w \in \mathrm{sp}\{v^1,\ldots,v^m\}$. Note that if $\mathrm{sp}\{v^1,\ldots,v^m\} \cong R^n$ then $A_v(w) = 0$.

Hence $0 = u\cdot Dg_0^{-1}D^2g_0(v,w) = (Dg_0u)\circ D^2g_0(v,w)$. If $\mathrm{sp}\{v^1,\ldots,v^m\} \cong R^n$ then $0 = Dg_0^{-1}D^2g_0(v,w)$. Since Dg_0 is an isometry, $0 = D^2g_0(v,w)$. q.e.d.

Lemma 5.3. Let g be a C^2 diffeomorphism of R^n. Let $E \subset R^n$. Suppose there exist points $x \in E$ converging to 0 where x is a limit point of E in the direction v_x. Suppose $\{v^1,\ldots,v^m\}$ are limit vectors of v_x as $x \to 0$. If $|1-\rho(Dg_x)| = O(\|x\|^2)$ for all $x \in E$ then $D^2g_0(v,w)\cdot Dg_0(u) = 0$ for $u, v, w \in \mathrm{sp}\{v^1,\ldots,v^m\}$. If $\mathrm{sp}\{v^1,\ldots,v^m\} \cong R^n$ then $D^2g_0(v,w) = 0$.

The proof is identical to that of Lemma 5.2.

6. Minimal isometry sets

DEFINITION

A set $E \subset M$ is *minimal* under f if it is invariant and contains no invariant subsets. A set E is an *isometry set* if Df is an isometry at each $x \in E$.

Theorem 6.1. *Let $f: M \to M$ be a C^3 mapping of a compact Riemannian 2-manifold M. Let $E \subset M$ be an isometry minimal set. Let $y, z \in E$ where z is in a local geodesic coordinate chart about y. Then $\| Df_y(z-y) + \frac{1}{2}D^2f_y(z-y)^2 \| \leqslant \|z-y\| + O(\|z-y\|^3)$.*

Here, "$(z-y)$" refers to the vector $u \in R^2$ such that $\exp_y(u) = z$; "Df_y" and "D^2f_y" are the first and second derivatives at 0 of the lift $\bar{f}: T_yM \to T_{fy}M$.

Notice that Taylor's theorem alone merely implies that $\|Df_y(z-y) + \frac{1}{2}D^2f_y(z-y)^2\| \leqslant \|z-y\| + O(\|z-y\|^2)$ if Df_y is an isometry. The proof of Theorem 2.1 uses the dynamics of f as well as the isometry condition on Df to sharpen the estimate.

Proof. We may assume that E is an infinite set, otherwise the result is trivial. It follows that E is perfect since it is an infinite minimal set.

First, suppose there exists a point $x \in E$ with sequences p and q in E approaching x from two directions.

Since f is C^1 (in fact C^3) and E is invariant, each point $f^n(x) = x_n$ has limit points in E from two directions. Since Df is an isometry at these points it follows from Lemma 5.1 and Lemma 5.2 that $D^2f = 0$ at x_n. Since E is minimal, $\{x_n\}$ is dense in E. Since $y \in E$, $D^2f_y = 0$. The estimate follows.

Now suppose there is no point in E with limit points from two directions. Then for each $x \in E$ there is a unique unit vector v_x along which points in E are converging towards x. Either v_x varies continuously or it does not. Suppose v_x is not continuous at p. Then there exist at least two distinct limits v' and v'' of v_x as $x \to p$. By

Lemma 5.3 we have $D^2f_p = 0$. By the smoothness of f this discontinuity is preserved, so $D^2f_x = 0$ for each orbit point $x = x_n$ and hence for each $x \in E$.

Suppose that v_x is a continuous function of x. We know by Lemma 5.2 only that $D^2f_x(v_x, v_x)\cdot Df_x(v_x) = 0$. We need to estimate the dot product $|D^2f_y(w, w)\cdot Df_y(w)|$ where $w = z - y$.

(6.2) *Let* $g: R^2 \to R^2$ *be* C^2. *If* $v\cdot D^2g_0(v, v) = 0$ *then* $|w\cdot D^2g_0(w, w)| \leqslant O\|w\|^3\cdot\sin(w, v)$.

Proof of 6.2. Assume $v = (1, 0)$ and $w = (1, y)$. Then $v\cdot D^2g_0(v, v) = (1, 0)\cdot(\partial^2 g_1/\partial x\partial x, \partial^2 g_2/\partial x\partial x) = 0$. Thus $\partial^2 g_1/\partial x\partial x = 0$. Using the Hessian matrix to express $D^2g_0(w, w)$ in coordinates we obtain

$$|w\cdot D^2g_0(w, w)| = \left|(1, y)\cdot\left(y\frac{\partial^2 g_1}{\partial y\partial x} + y\left(\frac{\partial^2 g_1}{\partial x\partial y} + y\frac{\partial^2 g_1}{\partial y^2}\right),\ \frac{\partial^2 g_2}{\partial x^2} + y\frac{\partial^2 g_2}{\partial y\partial x}\right.\right.$$
$$\left.\left. + y\left(\frac{\partial^2 g_2}{\partial x\partial y} + \frac{y\partial^2 g_1}{\partial y^2}\right)\right)\right|.$$

Therefore there exist constants $C > 0$ and $C' > 0$, depending on the second derivative of g, such that $|w\cdot D^2g_0(w, w)| \leqslant C\|y\| \leqslant C'|\sin(w, v)|$. The estimate 6.2 follows from the linearity of the dot product and the second derivative. Hence

$$(6.3)\qquad |D^2f_y(z - y)^2\cdot Df_y(z - y)| \leqslant O\|z - y\|^3\cdot|\sin(v_y, z - y)|.$$

We next estimate $|\sin(v_y, z - y)|$ for $y, z \in E$. The function v_y is continuous. It is uniformly continuous by compactness of M. Since v_y approximates $(z - y)/\|z - y\|$ it follows that $|\sin(v_y, z - y)| \leqslant 0\|z - y\|$. Hence, $|D^2f_y(z - y)^2\cdot Df_y(z - y)| \leqslant 0\|z - y\|^4$.

(6.4) *If* $a, b \in R^2$, $\|a\| \leqslant 1$, $|a\cdot b| \leqslant O\|a\|^4$ *and* $\|b\| \leqslant O\|a\|^2$ *then* $\|a + b\| \leqslant \|a\| + O\|a\|^3$.

Proof of 6.4. Let c be the unique vector such that $a\cdot(b - c) = c\cdot(b - c) = 0$. Then $|\cos\theta| = \|c\|/\|b\|$ where θ is the angle between a and b. We have $|a\cdot b| = \|a\|\,\|b\|\,|\cos\theta| \leqslant 0\|a\|^4$. Hence $\|a\|\,\|c\| \leqslant 0\|a\|^4$ and thus $\|c\| \leqslant 0\|a\|^3$. Note $\|b - c\| \leqslant \|b\| \leqslant 0\|a\|^2$. Since $a\cdot(b - c) = 0$ it follows that $\|a + (b - c)\| = \sqrt{(\|a\|^2 + \|b - c\|^2)} \leqslant \sqrt{(\|a\|^2 + C\|a\|^4)} \leqslant \|a\| + 0\|a\|^3$. Hence $\|a + b\| \leqslant \|a + (b - c)\| + \|c\| \leqslant \|a\| + 0\|a\|^3$.

By (6.3) and (6.4) we conclude

$$\|Df_y(z - y) + \tfrac{1}{2}D^2f_y(z - y)^2\| \leqslant \|z - y\| + 0(\|z - y\|^3). \qquad \text{q.e.d.}$$

COROLLARY 6.5

Let E *be an isometry minimal set of a* C^3 *mapping* f *of a compact Riemannian 2-manifold* M. *If* $y \in M$ *and* $z \in E$ *then*

$$\sum_{n=1}^{\infty} (d(f^n(y), f^n(z)))^2 = \infty$$

Proof. Let $y \in M$ and $z \in E$. Since E is compact, there exists a constant $\mu > 0$ and geodesic coordinate charts $U_{zn} = U_n$ based at $f_n(z) = z_n$ with radius $> \mu$. We can assume

that there exists a positive integer N such that y_n lies in U_n for $n \geqslant N$. Otherwise the result is immediate. Assume $n \geqslant N$.

By Taylor's theorem, since f is C^3 and E is compact, there exists a constant $A > 0$ such that in the coordinates of the chart U_n,

$$\frac{\| f(y_n) - (f(z_n) + Df_{z_n}(y_n - z_n) + \frac{1}{2}D^2 f_{z_n}(y_n - z_n)^2) \|}{\| y_n - z_n \|^3} < A.$$

But

$$\frac{|\, \| Df_{z_n}(y_n - z_n) + \frac{1}{2}D^2 f_{z_n}(y_n - z_n)^2 \| - \| y_n - z_n \| \,|}{\| y_n - z_n \|^3} < A'$$

Hence

$$\frac{\| f(y_n) - (f(z_n) + (y_n - z_n)) \|}{\| y_n - z_n \|^3} < A''.$$

Since $f(y_n) = y_{n+1}$ lies in U_{n+1} we may replace $f(y_n)$ by y_{n+1}. By the triangle inequality

$$\left| \frac{\| y_{n+1} - z_{n+1} \| - \| y_n - z_n \|}{\| y_n - z_n \|^3} \right| \leqslant \frac{\| y_{n+1} - (z_{n+1} - (y_n - z_n)) \|}{\| y_n - z_n \|^3} \leqslant A''.$$

6.6 *If a sequence of positive numbers $a_n \to 0$ satisfies*

$$\frac{|a_{a+1} - a_n|}{|a_n|^r} < A \text{ and } \frac{a_{a+1}}{a_n} > B > 0$$

then

$$\sum_{a=1}^{\infty} a_n^{r-1} = \infty.$$

Proof of (6.6). Since $a_n \to 0$ it follows that

$$\prod_{a=1}^{N} \frac{a_n}{a_{n+1}} = \frac{a_1}{a_N} \to \infty \quad \text{as } N \to \infty.$$

Hence

$$\sum_{n=1}^{\infty} \left| 1 - \frac{a_n}{a_{n+1}} \right| = \infty.$$

By hypothesis $(a_{n+1}/a_n) > B$ and $a_n^r > (|a_{n+1} - a_n|/A)$, so

$$a_n^{r-1} > \frac{1}{A}\left| \frac{a_{n+1}}{a_n} - 1 \right| = \frac{1}{A}\frac{a_{n+1}}{a_n}\left| 1 - \frac{a_n}{a_{n+1}} \right| > \frac{B}{A}\left| 1 - \frac{a_n}{a_{n+1}} \right|.$$

It follows from the comparison test that $\sum_{n=1}^{\infty} a_n^{r-1} = \infty$.

With (6.6) we may complete the proof of Corollary 6.5. Since Df is an isometry at z_n, there exists a constant $B > 0$ such that $\| y_{n+1} - z_{n+1} \| / \| y_n - z_n \| > B$. Let $a_n = \| y_n - z_n \|$ and $r = 3$. q.e.d.

COROLLARY 6.7

Let f be a C^3 diffeomorphism of a compact Riemannian 2-manifold M and $Q \subset M$ a minimal isometry set. If Q is a Denjoy Cantor set then Q is not square-rectifiable.

Proof. By definition, if Q is a Denjoy Cantor set in M there exists $h: Q \to S^1$ mapping Q onto a Denjoy Cantor set Γ in S^1. Let x_0 and y_0 be the inverse image of endpoints of an interval complementary to Γ in S^1. Since f preserves the order of Q the intervals (x_n, y_n) are each disjoint from Q and each other. By Corollary 6.5, $\sum d(x_n, y_n)^2 = \infty$. Thus Q is not square-rectifiable. q.e.d.

DEFINITION

A Jordan curve Q in a Riemannian two-manifold is a *quasi-circle* if there exists a positive constant K such that if $x, y \in Q$ then one of the arcs of Q connecting x and y is contained in a disk of radius $Kd(x, y)$.

PROPOSITION 6.8

A quasi-circle Q is square-rectifiable.

Proof. Let $x_1 < x_2 < \cdots$ be a partition of Q.

Let $a_n = d(x_n, x_{n+1})$ and B_n the disk of radius $a_n/8K$ centered at x_n where K is the quasi-constant for Q. The result follows from compactness of Q if the B_n are disjoint. Suppose $B_n \cap B_m \neq \emptyset$. Then $d(x_n, x_m) < (a_n + a_m)/8K \leqslant a_n/4K$, assuming $a_m \leqslant a_n$. Since Q is a quasi-circle, one of the arcs connecting x_n and x_m is contained in a disk of radius $a_n/4$. Let z be a point in this arc equidistant from x_n and x_m. Then $d(x_n, z) = d(x_m, z) < a_n/2$. Thus $a_n = d(x_n, x_m) \leqslant d(x_n, z) + d(z_n, x_m) < a_n$. q.e.d.

COROLLARY 6.9

Let f be a C^3 diffeomorphism of a compact Riemannian 2-manifold M and $Q \subset M$ a minimal isometry set. If Q is a Denjoy Cantor set then Q is not square-rectifiable.

References

[1] Denjoy A, Sur les courbes définies par les equations differentielles a la surface du tore, *J. Math. Pure Appl.* **11** (1932) 333–375

[2] Hall G R, A C^∞ Denjoy counter-example, *Ergodic Theory and Dynamical Systems*, **1** (1981) 261–272

[3] Harrison J, Continued fractals and the Seifert conjecture, *Bull. Am. Math. Soc.* **13** (1985) 147–153

[4] Harrison J, C^2 counter-examples to the Seifert conjecture, *Topology* **27** (1988) 49–78

[5] Harrison J, Denjoy fractals, *Topology* **28** (1989) 59–80

[6] Herman M, Sur les courbes invariants par les diffeomorphismes de l'anneau, *Asterisque.* **1** (1983) 103–104

[7] Katok A, *Periodic and quasi-periodic orbits for twist maps.* Berlin Springer–Verlag (1983)

[8] Mather J, Existence of quasi-periodic orbits for twist homeomorphisms of the annulus, *Topology* **21** (1982) 457–467

[9] Moser J, *Stable and random motions in dynamical systems, Ann. Math. Stud.* (Princeton: University Press) (1973)

[10] Sullivan D, Conformal dynamical systems (preprint)

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 2, August 1989, pp. 123–126.

On the gap between two classes of analytic functions

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Abstract. Two large classes of analytic functions are defined, so that one contains the other. Sharp coefficient bounds for quadratic polynomials falling in the gap between these two classes are given.

Keywords. Analytic function; Kaplan classes; Hadamard product.

1. Introduction

The Kaplan class $K(\alpha,\beta)$ consists of those functions $f(z) = 1 + a_1 z + \cdots$, analytic and non-zero in the open unit disk $U = \{z: |z| < 1\}$ such that

$$-\alpha\pi \leqslant \int_{\theta_1}^{\theta_2} \{\mathrm{Re}(zf'/f) - \tfrac{1}{2}(\alpha - \beta)\}\, \mathrm{d}\theta \leqslant \beta\pi,$$

where $z = r\exp(i\theta)$, $0 < r < 1$, $\theta_1 < \theta_2 < \theta_1 + 2\pi$, $\alpha \geqslant 0$ and $\beta \geqslant 0$. ([5]).

The class $T(\alpha,\beta)$ consists of those functions $\phi(z) = 1 + b_1 z + \cdots$, analytic and non-zero in U such that for $|u| \leqslant 1$ and $|v| \leqslant 1$, $\phi(z)*\{(1+uz)^t(1+vz)^{\alpha-t}/(1-z)^\beta\} \neq 0$ in U where $\alpha \geqslant 0$, $\beta \geqslant 0$, t is the integer part of α and the operation $*$ is the Hadamard product or convolution of two power series. ([5]).

The dual of $T(\alpha,\beta)$, denoted by $T(\alpha,\beta)^*$, consists of those functions $g(z) = 1 + c_1 z + \cdots$, analytic and non-zero in U such that $(g*\phi)(z) \neq 0$ in U for all $\phi \in T(\alpha,\beta)$ where $\alpha \geqslant 0$ and $\beta \geqslant 0$. ([5]).

Several well-known classes of univalent and analytic functions are closely related to $K(\alpha,\beta)$, $T(\alpha,\beta)$ and its dual. For instance, f is close-to-convex of order $\alpha \geqslant 0$, ([5] p. 224), if and only if $f' \in K(\alpha, \alpha+2)$, and f is prestarlike of order $\alpha \leqslant 1$, ([4] p. 47), if and only if $f/z \in T(1, 3-2\alpha)$. For more references see [1, 4, 5, 6, 7].

It is shown in [5] that for $\alpha \geqslant 1$ and $\beta \geqslant 1$, $K(\alpha,\beta)$ is contained in the dual of $T(\alpha,\beta)$. The gap between $K(\alpha,\beta)$ and $T(\alpha,\beta)^*$ seems to be very large, and little is known about it.

In the following theorem we give an example of a polynomial having all its zeros on $|z| = 1$ that is in $T(1,\beta)^* - K(1,\beta)$; $\beta \geqslant 1$. We note that though the example is in the form $p(z) = 1 + \gamma z + z^2$ with bounds on $|\gamma|$, it justifies the existence of functions of the form $f(z) = 1 + az + bz^2$ having all their zeros on $|z| = 1$ and belonging to $T(1,\beta)^* - K(1,\beta)$ with analogous restrictions on $|a|$. To see this, write $f(z) = 1 + \gamma\exp(i\theta)z + \exp(i\phi)z^2$ and let $g(z) = f(z\exp(-i\phi/2)) = 1 + \gamma\exp(i(\theta - \phi/2)\, z + z^2$ where γ is real. Since $f(z)$ vanishes only on $|z| = 1$, $g(z)$ must also vanish only on $|z| = 1$. Using the fact that $1 + Az + z^2$ has all its zeros on $|z| = 1$ only when A is real,

we conclude that $\gamma \exp(i(\theta-\phi/2))$ must be real, that is, $\phi = 2n\pi + 2\theta$ where $n=0, \pm 1, \ldots$. Now comparing $p(z)$, $f(z)$ and $g(z)$ we observe that $|a| = |\gamma \exp(i\theta)| = |\gamma|$.

Theorem. *Let $\beta \geqslant 1$ and let γ be real. Then $p(z) = 1 + \gamma z + z^2$, $|z| < 1$ is in the dual of $T(1,\beta)$ but not in $K(1,\beta)$ if and only if $2\cos(\pi/(1+\beta)\} \leqslant |\gamma| \leqslant [4(\beta^2-1)/\beta^2]^{\frac{1}{2}}$.*

Proof. The left-hand-bound for $|\gamma|$ comes from the following Lemma A, which is a special case of Lemma 3 in [1].

Lemma A. Let $\beta \geqslant 1$ and let γ be real. Then $1 + \gamma z + z^2$ is in $K(1,\beta)$ if and only if $|\gamma| \leqslant 2\cos\{\pi/(1+\beta)\}$.

To prove the right-hand-bound for $|\gamma|$, we shall need the following two known lemmas, the first of which is a consequence of Cohn's rule ([3] or [8] p. 31).

Lemma B. Suppose that $p(z) = 1 + az + bz^2$ is such that $|b| < 1$. Then $p(z) \neq 0$ in $|z| < 1$ if and only if $p(z) - bz^2\overline{p(1/\bar{z})} \neq 0$ in $|z| < 1$.

Lemma C. If $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is in $T(1,\beta)$ for $\beta \geqslant 0$, then $|c_2 - c_1^2| \leqslant \{1 - |c_1|^2\}/\beta$. Also $|c_1| \leqslant 1$ and $|c_2| \leqslant 1$.

Lemma C is a modification of a result by Ruscheweyh ([4] p. 54) using the relation between préstar like functions and the class $T(1,\beta)$.

Now we shall justify the right-hand-bound for $|\gamma|$ in the theorem, where γ is real and $\beta \geqslant 1$. We have $p(z) = 1 + \gamma z + z^2$ is in $T(1,\beta)^*$ if and only if for any $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ in $T(1,\beta)$,

$$\psi(z) = (p*\phi)(z) = 1 + \gamma c_1 z + c_2 z^2 \neq 0 \text{ in } U. \tag{1}$$

Note that since $p(z) = 1 + \gamma z + z^2 \in T(1,\beta)^*$, $p(z) \neq 0$ in U and so $|\gamma| \leqslant 2$. Since $\phi(z) \in T(1,\beta)$ if and only if $\phi(z\exp(i\theta)) \in T(1,\beta)$ for real θ, without loss of generality, we assume that $c_1 \geqslant 0$. If $c_1 = 0$, then (1) holds for all $|\gamma| \leqslant 2$ and $|c_2| \leqslant 1$. If $c_1 = 1$, then by Lemma C, $c_2 = 1$ and so $\psi(z)$ reduces to $p(z) = 1 + \gamma z + z^2$, which is non-zero in U for all $|\gamma| \leqslant 2$. Furthermore, if $|c_2| = 1$, then for $c_2 = \exp(iv)$, where v is real, we observe that $\psi(z) = 1 + \gamma c_1 z + \exp(iv)\, z^2 \neq 0$ in U for all $|\gamma| \leqslant 2$ and $0 < c_1 < 1$.

Since for the cases $c_1 = 0$, $c_1 = 1$ and $|c_2| = 1$, (1) leads to $|\gamma| \leqslant 2$ which holds when $|\gamma| \leqslant \sqrt{4(\beta^2-1)/\beta^2}$, we are left with the remaining cases $0 < c_1 < 1$ and $|c_2| < 1$. By applying Lemma B to (1) we obtain that $\psi(z) \neq 0$ in U if and only if $\psi(z) - c_2 z^2 \overline{\psi(1/\bar{z})} = 1 - |c_2|^2 + \gamma c_1(1 - c_2)\, z \neq 0$ in U or if and only if

$$|\gamma| \leqslant \{(1 - |c_2|^2)/(c_1|1 - c_2|)\}. \tag{2}$$

The inequality in Lemma C can be written as $c_2 = c_1^2 + \{\xi(1 - c_1^2)\}/\beta$ where $|\xi| \leqslant 1$. We substitute this in (2) to obtain

$$|\gamma| \leqslant \frac{1 - |c_2|^2}{c_1|1 - c_2|} = \frac{1 - \left|c_1^2 + \dfrac{1}{\beta}\xi(1 - c_1^2)\right|^2}{c_1\left|1 - c_1^2 - \dfrac{1}{\beta}\xi(1 - c_1^2)\right|}$$

$$= \frac{1 - \left\{c_1^4 + \dfrac{|\xi|^2}{\beta^2}(1-c_1^2)^2 + \dfrac{2}{\beta}c_1^2(1-c_1^2)\operatorname{Re}\xi\right\}}{c_1(1-c_1^2)\left|1 - \dfrac{1}{\beta}\xi\right|}$$

$$= \frac{1 + c_1^2 - \dfrac{|\xi|^2}{\beta^2}(1-c_1^2) - \dfrac{2}{\beta}c_1^2\operatorname{Re}\xi}{c_1\left\{1 + \dfrac{|\xi|^2}{\beta^2} - \dfrac{2}{\beta}\operatorname{Re}\xi\right\}^{1/2}}$$

$$= \frac{1 - \dfrac{|\xi|^2}{\beta^2} + c_1^2\left\{1 + \dfrac{|\xi|^2}{\beta^2} - \dfrac{2}{\beta}\operatorname{Re}\xi\right\}}{c_1\left\{1 + \dfrac{|\xi|^2}{\beta^2} - \dfrac{2}{\beta}\operatorname{Re}\xi\right\}^{1/2}}. \tag{3}$$

Letting

$$X = c_1\left\{1 + \frac{|\xi|^2}{\beta^2} - \frac{2}{\beta}\operatorname{Re}\xi\right\}^{1/2}$$

and $Y = 1 - (|\xi|^2/\beta^2)$ in (3) we obtain

$$|\gamma| \leqslant (Y + X^2)/X \text{ or } X^2 - |\gamma|X + Y \geqslant 0. \tag{4}$$

(4) is true if $|\gamma| \leqslant 2\sqrt{Y} = 2\{1 - \beta^{-2}|\xi|^2\}^{1/2}$ and so for $|\xi| = 1$, we obtain

$$|\gamma| \leqslant \sqrt{4(\beta^2-1)/\beta^2}. \tag{5}$$

For the second part of the proof, assume that the inequality (5) is true. We wish to show that $p(z) = 1 + \gamma z + z^2$ is in $T(1,\beta)^*$. In other words, we need to show that $\psi(z) = (p * \phi)(z) = 1 + \gamma c_1 z + c_2 z^2 \neq 0$ in U for all $\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots$ in $T(1,\beta)$. Suppose that there exists z_1 in U such that

$$\psi(z_1) = 1 + c_1\gamma z_1 + c_2 z_1^2 = 0. \tag{6}$$

We consider the following two cases for c_2 in (6). If $|c_2| = 1$, then $|\gamma| > 2$ since $|c_1| \leqslant 1$ and $|z_1| < 1$. If $|c_2| < 1$, without loss of generality we assume that $c_1 \geqslant 0$ and then by Lemma B, $\psi(z_1) = 0$ if and only if $1 - |c_2|^2 + c_1\gamma(1 - c_2)\, z_1 = 0$ or if and only if

$$|\gamma| > \{1 - |c_2|^2\}/\{c_1|1 - c_2|\}. \tag{7}$$

For $|\xi| \leqslant 1$, we obtain from $c_2 = c_1^2 + \{\xi(1-c_1^2)\}/\beta$ and (7) that $|\gamma| > (X^2 + Y)/X$ where X and Y are same as in (4). Then $X^2 - |\gamma|X + Y < 0$ or $(X - \frac{1}{2}|\gamma|)^2 + Y - \frac{1}{4}|\gamma|^2 < 0$. This is true only if $Y - \frac{1}{4}|\gamma|^2 < 0$ or $|\gamma| > 2\sqrt{Y}$ and so substituting for Y we obtain $|\gamma| > \sqrt{4(\beta^2-1)/\beta^2}$.

In either case, we see that the bound obtained for γ contradicts (5). Thus $\psi(z) \neq 0$ in U and so $p(z)$ is in the dual of $T(1,\beta)$. This completes the proof of the theorem.

2. Discussion and an open problem

Using the fact that $K(1,3)$ is the class of the derivatives of close-to-convex functions defined by Kaplan [2], we deduce from our theorem that for real γ, $z+\frac{1}{2}\gamma z^2+\frac{1}{3}z^3$ is not close-to-convex, while its derivative is in the dual of $T(1,3)$ if and only if $\sqrt{2}\leqslant|\gamma|\leqslant\sqrt{32/9}$. The special case $z+\sqrt{8/9}z^2+\frac{1}{3}z^3$ was studied earlier by Ruscheweyh ([4] pp. 76–78) using different notations.

Apart from the case $\alpha=1$, it is not easy to work with $T(\alpha,\beta)$ and its dual and so a generalization of our theorem is open.

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References

[1] Jahangiri M, A gap condition for the zeros of certain polynomials in Kaplan classes $K(\alpha,\beta)$, *Mathematika* **34** (1987) 53–63

[2] Kaplan W, Close-to-convex schlicht functions, *Michigan Math. J.* **1** (1952) 169–185

[3] Marden M, The geometry of the zeros of a polynomial in a complex variable, *Am. Math. Soc.* Math. Surveys No. 3 (second edition) (1966)

[4] Ruscheweyh St, Convolutions in geometric function theory In: *Seminaire de Mathematiques superieures* De L'Universite De Montreal Les Press (1982)

[5] Sheil-Small T B, The Hadamard product and linear transformations of classes of analytic functions, *J. Analyse Math.* **34** (1978) 204–239

[6] Sheil-Small T B, Coefficients and integral means of some classes of analytic functions, *Proc. Am. Math. Soc.* **88** (1983) 275–282

[7] Sheil-Small T B, Some remarks on Bazilevic functions, *J. Anal. Math.* **43** (1983/84) 1–11

[8] Suffridge T J, Polynomials in function theory, *Contemp. Math.* **38** (1985) 31–42

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 2, August 1989, pp. 127–146.

Discrete subgroups of algebraic groups over local fields of positive characteristics

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Abstract. It is shown in this paper that if G is the group of k-points of a semisimple algebraic group G over a local field k of positive characteristic such that all its k-simple factors are of k-rank 1 and $\Gamma \subset G$ is a non-cocompact irreducible lattice then Γ admits a fundamental domain which is a union of translates of Siegel domains. As a consequence we deduce that if G has more than one simple factor, then Γ is finitely generated and by a theorem due to Venkataramana, it is arithmetic.

Keywords. Discrete subgroups; algebraic groups; local fields; Siegel domains; fundamental domains; positive characteristics.

1. Introduction

Let k_i, $i \in I$, be local fields of characteristic $p > 0$ with I a finite set. For $i \in I$, let $\mathbf{G}_i$ be an algebraic group defined, absolutely almost simple and of *rank* 1 over k_i. Let $\mathbf{G}_i(k) = G_i$ and $G = \prod_{i \in I} G_i$. Let $\Gamma \subset G$ be a discrete subgroup such that the volume of G/Γ with respect to the measure on G/Γ induced by a (bi-invariant) Haar measure μ on G is finite. Our aim in this paper is to exhibit for such a Γ a "good" fundamental domain in G. The existence of the good fundamental domain leads to the following result: if $|I| \geqslant 2$ and Γ is irreducible, then Γ is finitely generated. (Γ is irreducible if for every $i \in I$, the projection of G on G_i is injective when restricted to Γ). The existence of the fundamental domain is proved along the same lines as in characteristic 0 (see Raghunathan [6]) but this requires that we extend some results of Kazdan-Margulis [5]. These last results cannot be extended in the form given in [5] unless p is a good characteristic for G_i for all $i \in I$. We prove a modified version of the main lemma of that paper a version which is weaker but somewhat more delicate to prove. (The stronger statement that holds in characteristic 0 appears to be false in bad positive characteristics.) The modified version being weaker necessitates somewhat more subtle arguments than those given in [5] to draw further conclusions towards the construction of the fundamental domain. However one obtains as a straightforward consequence the following: There is a positive constant $C = C(\mu)$ such that $\mathrm{vol}(G/\Gamma) > C$ (in the measure determined by a Haar measure on G—the constant is independent of Γ).

The fundamental domain constructed is a union of a finite number of translates of a "Siegel domain" and is thus an extension of the corresponding result for arithmetic

groups (Behr [1], Harder [4]). The standard properties that hold for these fundamental domains in the case of arithmetic groups carries over to the possibly non-arithmetic case as well but we have not elaborated on this here. We note that our construction is available only when G is a product of groups of rational points of rank 1 groups over local fields. However when G has even one factor of rank $\geqslant 2$, a theorem due to Venkataramana [9] assures us that an irreducible Γ is necessarily arithmetic (and we can appeal to Harder [4] for the existence of good fundamental domains). Actually Venkataramana proves that an irreducible Γ is arithmetic even when all the factors are of rank 1 if $|I| \geqslant 2$ and provided that Γ *is finitely generated.* Our results guarantee that Γ is indeed finitely generated and thus the assumption needed for proving arithmeticity is indeed always valid. It must however be noted that the finite generation of Γ is deduced as a consequence of the existence of a good fundamental domain and thus the proof of the arithmeticity uses the existence of a good fundamental domain.

Throughout this paper we deal only with the case when all the G_i are *adjoint groups.* It is easy to deduce the general case by using the isogeny of any group onto its adjoint group. We have not spelt out the details; even the formulations are confined to the adjoint group.

Although our main interest is in lattices we formulate our results in an apparently more general context. A discrete subgroup Γ of a locally compact group F is a *L-subgroup* if for every neighbourhood V of 1 in F, there is a compact set $K(V) \subset F$ such that $K(V)\Gamma$ contains $\{g \in F \mid g\Gamma g^{-1} \cap V = \{1\}\}$. Any lattice in a locally compact group is a L-subgroup (chapter I, [6]). We prove our results first for *irreducible* L-subgroups in G: $\Gamma \subset G$ is *irreducible* if the restriction to Γ of the cartesian projection of G on G_i is injective for every i. We then show that if $\Gamma \subset G$ is an irreducible L-subgroup it has a suitable density property. From this density property, we deduce the following. If $\Gamma \subset G$ is any L-subgroup, there is a partition $I = \bigcup_{1 \leqslant \alpha \leqslant r} I_\alpha$ of I such that the following holds: let $H_\alpha = \prod_{i \in I_\alpha} G_i$ and Γ_α the projection of Γ on H, then Γ_α is an irreducible L-subgroup and Γ has finite index in $\prod_{1 \leqslant \alpha \leqslant r} \Gamma_\alpha$. (It turns out after the construction of the fundamental domain that any L-subgroup in G is indeed a lattice. It is of some mild interest to know if this is true even when G has factors which are of higher rank.)

I would like to thank Venkataramana in collaboration with whom I had earlier made some progress on the questions treated here. My thanks are also due to Margulis for his interest in this work and to Harder for making it possible for me to meet and talk to Margulis in *Bonn.*

We use results from the theory of algebraic groups freely without citing references. Much of the background material needed is to be found in Borel-Tits [2] and some use is made also of the classification results due to Tits [8].

2. Some lemmas on k-rank 1 algebraic groups

2.1 Let k be a local field of characteristic $p > 0$ and $\mathbf{G}$ a connected absolutely simple k-algebraic group of adjoint type of k-rank 1. We denote by G the locally compact group $\mathbf{G}(k)$ of k points of $\mathbf{G}$. In the sequel algebraic subgroups of $\mathbf{G}$ are denoted by Roman capital letters in bold face their k-points by corresponding plain types, their Lie algebras by lower case gothic letters in bold face while the k-points of these

Lie algebras are denoted by the gothic letters in plain type. In particular the Lie algebra of **G** is denoted $\mathfrak{g}$. We also identify Lie algebras of algebraic subgroups of **G** with the corresponding Lie subalgebras of $\mathfrak{g}$. Let **T** be a maximal k-split torus in **G**; by assumption $\dim \mathbf{T} = 1$ so that the character group $X(\mathbf{T})$ of **T** is isomorphic to $\mathbb{Z}$. We fix a generator α of $X(\mathbf{T})$ and define a character $\phi \in X(\mathbf{T})$ to be *positive* if $\phi = r\alpha$ with $r > 0$; also $\phi < 0$ if $-\phi > 0$. Let Φ denote the k-root system of **G** with respect to **T**, $\Phi^+ = \{\phi \in \Phi \mid \phi > 0\}$ and $\Phi^- = \{\phi \in \Phi \mid \phi < 0\}$. Then Φ^+ is of the form $\{\alpha\}$ or $\{\alpha, 2\alpha\}$ for a unique root $\alpha \in \Phi$. The root α is the unique simple root of **G** with respect to **T** and the ordering defined above. Let **Z** (resp. **N**) denote the centraliser (resp. normaliser) of **T** in **G**. We denote by **U** (resp. **V**) the unique k-split maximal unipotent subgroup of **G** normalized by T and having for its Lie algebra $\mathfrak{u}$(resp $\mathfrak{v}$) the sum $\bigsqcup_{\phi \in \Phi^+} \mathfrak{g}^{\phi}$ (resp $\bigsqcup_{\phi \in \Phi^-} \mathfrak{g}^{\phi}$) of the root-spaces of **G** (w.r.t. **T**) corresponding to the positive roots. We have then $\mathfrak{g} = \mathfrak{u} + \mathfrak{z} + \mathfrak{v}$. Let **P** (resp. **Q**) be the k-group $\mathbf{Z} \cdot \mathbf{U}$ (resp. $\mathbf{Z} \cdot \mathbf{V}$); then **P** and **Q** are minimal k-parabolic subgroups of **G**. If $2\alpha \in \Phi^+$, the centre $\mathbf{U}'$(resp. $\mathbf{V}'$) is a **Z**-stable k-split subgroup with Lie algebra $\mathfrak{u}' = \mathfrak{g}^{2\alpha}$(resp. $\mathfrak{v}' = \mathfrak{g}^{-2\alpha}$). Moreover $\mathbf{U}'$ (resp. $\mathbf{V}'$) is isomorphic to $\mathfrak{g}^{2\alpha}$ (resp. $\mathfrak{g}^{-2\alpha}$) as a k-vector space by an isomorphism compatible with the action of **Z**. The group $\mathbf{U}/\mathbf{U}'$ (resp. $\mathbf{V}/\mathbf{V}'$) is also k-isomorphic to a vector space viz $\mathfrak{u}/\mathfrak{u}' \simeq \mathfrak{g}^{\alpha}$ (resp $\mathfrak{v}/\mathfrak{v}'$ $\mathfrak{g}^{-\alpha}$) through an isomorphism again compatible with **Z**-action. Since $\mathbf{U}/\mathbf{U}'$ is affine the natural map: $\mathbf{U} \to \mathbf{U}/\mathbf{U}'$ a $\mathbf{U}'$-fibration is necessarily trivial. Hence we can find a section θ: $\mathbf{U}/\mathbf{U}' \to \mathbf{U}$ to ω defined over k. θ can in fact be chosen to be compatible with the action of **T** on $\mathbf{U}/\mathbf{U}'$ and $\mathbf{U}$*. We assume that θ is chosen in this fashion; in particular this means that $\theta(1) = 1$. Let $\sigma\colon U \to U$ be the morphism $\theta \cdot \omega$ and define $\tau\colon \mathbf{U} \to \mathbf{U}'$ by setting $u = \sigma(u).\ \tau(u)$. Note that if $2\alpha \notin \Phi$, $\tau(u) = u$. We denote by G' the group generated by U' and $\mathbf{V}'$. Then $\mathbf{G}'$ is an absolutely simple k-group which contains **T**, is of k-rank 1 and is *simply connected* if $2\ \alpha \in \Phi^+$. Let ρ: $\mathbf{G} \to GL(\mathfrak{g})$ (resp.) denote the adjoint representation of **G** (resp. $\mathfrak{g}$) on $\mathfrak{g}$. Then $\dot{\rho}(\mathfrak{z})$ leaves $\mathfrak{u}$ stable and we denote by $\dot{\rho}^+$ the representation of $\mathfrak{z}$ on $\mathfrak{u}$ obtained by restring $\dot{\rho}$. It is known – and not difficult to show – that $\dot{\rho}^+$ is *faithful*. The product map $\beta\colon \mathbf{U} \times \mathbf{Z} \times \mathbf{V} \to \mathbf{G}$ defined by $\beta(u, z, v) = u \cdot z \cdot v$, $u \in \mathbf{U}$, $z \in \mathbf{Z}$, $v \in \mathbf{V}$ is a k-isomorphism of $U \times Z \times V$ onto an open subset $\mathbf{\Omega}$ of **G**. We define morphisms $u\colon \mathbf{\Omega} \to \mathbf{U}$, $z\colon \mathbf{\Omega} \to \mathbf{Z}$, $v\colon \mathbf{\Omega} \to \mathbf{V}$ by setting $\beta^{-1}(x) = (u(x), z(x), v(x))$ for $x \in \mathbf{\Omega}$. Evidently one has $x = u(x) \cdot z(x) v(x)$ for $x \in \mathbf{\Omega}$. We set $\Omega = \mathbf{\Omega}(\mathbf{k})$; then Ω is an open (dense) subset of G. The results summarized and notations introduced above will be used freely in the sequel.

2.2 The group **Z** is reductive and its commutator subgroup **M** is *anisotropic* over k. It follows that M $(= \mathbf{M}(k))$ is *compact* and (hence) Z has a *unique* maximal compact (open) subgroup Z. Now it is well-known that Z has index 2 in N (and $N/Z \simeq \mathbf{N}/\mathbf{Z}$). Let ν be any element in $N \backslash Z$ fixed once and for all. Evidently ν normalizes Z so that $\mathsf{N} = \mathsf{Z} \cup \nu\mathsf{Z}$ is a maximal compact subgroup in N. Let Λ be the maximal compact subring in k. Then $\mathfrak{g}$ admits a Λ-*free* submodule L with the following properties:

(i) $\mathfrak{g}$ is the k-span of L
(ii) L is N-stable
(iii) $L = L \cap \mathfrak{u} + L \cap \mathfrak{z} + L \cap \mathfrak{v}$.

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* See Appendix

(To secure the last condition – in case the residue field of k is small – one argues by adding onto N the group of elements of finite order in $\mathbf{T}(k')$ for an unramified extension k' and looking at the action of this bigger compact group on $\mathfrak{g}(k')$.) The Λ-module L enables us to define a compact open subgroup G of G: we set $\mathrm{G} = \{x \in G \mid x(L) = L\}$ (here and often in the sequel we identify x in G with its image $\rho(x) \in GL(\mathfrak{g})$). Clearly $\mathrm{G} = G \cap GL(L)$ (where $GL(L) = \{x \in GL(V) \mid x(L) = L\}$; $GL(L)$ is a compact open subgroups of $GL(\mathfrak{g})$). For an integer $r > 0$, let

$$GL(L)(r) = \{x \in GL(L) \mid (x-1)(L) \subset \pi^r L\},$$

where π is a uniformising parameter. If $e_1, \ldots, e_n$ is a Λ-basis of L and we identify $GL(\mathfrak{g})$ with $GL(n,k)$ through this basis, one has for an integer $r \geqslant 0$,

$$GL(L)(r) = \{x \in GL(n,\Lambda) \mid x \equiv 1 \,(\mathrm{mod}\, \pi^r)\}.$$

Let $| \ |: k \to \mathbb{R}^+$ be the absolute value on k given by $|x| = p^{-r}$ where $x = \pi^r \cdot u$, u a unit in Λ and $r \in \mathbb{Z}$ and $|0| = 0$. As usual we define $\|X\|$ for a matrix $X \in M(n,k)$ by setting $\|X\| = \max(|X_{ij}|, 1 \leqslant i,j \leqslant n)$. Using this norm we define a metric on $GL(\mathfrak{g})$ which is left-translation invariant as follows: let $g, h \in GL(\mathfrak{g})$ then $d(g,h) = p$ if $g^{-1}h \notin GL(L)$; if $g^{-1}h \in GL(L)$, we set $d(g,h) = \inf\{p^{-r} \mid g^{-1}h \in GL(L)(r)\}$. The family $GL(L)(r)$, $r \in \mathbb{Z}^+$ is a fundamental system of neighbourhoods of 1 in $GL(\mathfrak{g})$ so that the metric d defined above is compatible with the topology on $GL(\mathfrak{g})$. We obtain a left translation invariant metric on G by restricting this metric to it. We also set for $g \in GL(g)$, $|g| = d(1,g)$; it is easy to see then that if $g \in GL(L)(r) \backslash GL(L)(r+1)$ for an integer $r \geqslant 0$, then $|g| = p^{-r}$. In other words $GL(L)(r) = \{g \in GL(g) \mid |g| \leqslant p^{-r}\}$. Also for $g \in GL(L)$, $|g| = \|g-1\| \,(= \|g^{-1} - 1\|)$. As is well known we have $[GL(L)(r), GL(L)(s)] \subset GL(L)(r+s)$ for integers $r, s \geqslant 0$. In particular $GL(L)(r)$ is normal in $GL(L)$. Let $\mathrm{G}(r) = GL(L)(r) \cap G$, $r \in \mathbb{Z}^+$; then $\mathrm{G}(r)$ are compact open normal subgroups of G and $[\mathrm{G}(r), \mathrm{G}(s)] \subset \mathrm{G}(r+s)$ and for $g \in \mathrm{G}$, $|g| = \inf\{p^{-r} \mid g \in \mathrm{G}(r)\}$. We will now establish a series of lemmas using the notions in 2·1 and 2·2.

Lemma 2.3. $\mathrm{G}(1) \subset \Omega$ *and for* $x \in \mathrm{G}(1)$, *we have* $|x| = \max(|u(x)|, |z(x)|, |v(x)|)$. (*Consequently*) *if* $x, y \in \mathrm{G}(1)$ *and* $x = y (\mathrm{mod}\, \mathrm{G}(r))$ *for some* $r \geqslant 0$, $u(x) = u(y) (\mathrm{mod}\, \mathrm{G}(r))$; $z(x) = z(y) (\mathrm{mod}\, \mathrm{G}(r))$ *and* $v(x) = v(y) (\mathrm{mod}\, \mathrm{G}(r))$.

Proof. One has a more general fact. Let $\tilde{\mathbf{U}}$ (resp. $\tilde{\mathbf{V}}$) denote the subgroup of $GL(\mathfrak{g})$ stabilising $\mathfrak{u}$ and $\mathfrak{u} + \mathfrak{z}$ (resp. $\mathfrak{v}$ and $\mathfrak{z} + \mathfrak{v}$) and acting trivially on $\mathfrak{u}$ (resp. $\mathfrak{v}$), $(\mathfrak{u} + \mathfrak{z})/\mathfrak{u}$ (resp. $(\mathfrak{z} + \mathfrak{v})/\mathfrak{v}$) and $\mathfrak{g}/(\mathfrak{u} + \mathfrak{z})$ (resp. $\mathfrak{g}/(\mathfrak{z} + \mathfrak{v})$). Let $\tilde{\mathbf{Z}}$ be the group $\{g \in GL(\mathfrak{g}) \mid g(\mathfrak{u}) = \mathfrak{u}, g(\mathfrak{z}) = \mathfrak{z}$ and $g(\mathfrak{v}) = \mathfrak{v}\}$. (Then $\mathbf{U} \subset \tilde{\mathbf{U}}$, $\mathbf{V} \subset \tilde{\mathbf{V}}$ and $\mathbf{Z} \subset \tilde{\mathbf{Z}}$.) The morphism $(u,z,v) \to u.z.v.$ of $\tilde{\mathbf{U}} \times \tilde{\mathbf{Z}} \times \tilde{\mathbf{V}}$ in $GL(\mathfrak{g})$ is an isomorphism onto a Zariski open set $\tilde{\mathbf{\Omega}}$ and $GL(L)(1) \subset \tilde{\Omega} = \tilde{\mathbf{\Omega}}(k)$; and if $x = u.z.v$, $u \in \tilde{U} = \tilde{\mathbf{U}}(k)$, $z \in \tilde{Z} = \tilde{\mathbf{Z}}(k)$ and $v \in \tilde{V} = \tilde{\mathbf{V}}(k)$ with $x \in GL(L)(1)$, then one has $|x| = \max(|u|, |z|, |v|)$; this can be checked by explicit matrix computations using the representation of matrices in $\mathrm{End}(\mathfrak{g})$ by blocks corresponding to the direct-sum decomposition $\mathfrak{g} = \mathfrak{u} + \mathfrak{z} + \mathfrak{v}$ (note that our choice of L ensured that $L = L \cap \mathfrak{u} + L \cap \mathfrak{z} + L \cap \mathfrak{v}$).

Lemma 2.4. (i) *If* $g \in \mathrm{G}(1)$ *is such that* $|u(g)| \geqslant \max(|z(g)|, |v(g)|)$ *and* x *is any element of* $\mathrm{G}(1)$, *then* $|u(xgx^{-1})| \geqslant \max(|z(ugx^{-1})|, |v(xgx^{-1})|)$ *and the latter inequality is strict if and only if the former is.*

(ii) *If* $g \in \mathrm{G}\,(1)$ *and* $x \in \mathrm{G}\,(1) \cap P$, $|v(xgx^{-1})| = |v(x)|$; *if* $|z(g)| \geqslant |v(g)|$ *one has also* $|z(xgx^{-1})| = |z(x)|$. *If* $x \in \mathrm{G}\,(1) \cap U$ *and* $|u(g)| \geqslant |z(g)| \geqslant v(g)$, *then* $|u(x\hat{g}x^{-1})| = |u(g)|$, $|z(xgx^{-1})| = |z(g)|$ *and* $|v|xgx^{-1})| = |v(g)|$.

Proof. We have $xgx^{-1} = u(g)\cdot u(g)^{-1}xu(g)x^{-1}\cdot x\xi x^{-1}\xi^{-1}\cdot\xi$ where $\xi = z(g)\cdot v(g) \in Q$. Now $|g| = p^{-r}$ we have $u(g) \in \mathrm{G}(r)\backslash G(r+1)$ while $\xi \in \mathrm{G}(r)$. It follows that $u(g)^{-1}xu(g)\cdot x^{-1}\cdot x\xi x^{-1}\xi^{-1} \in \mathrm{G}(r+1)$. Since $\xi \in Q$, $u(xgx^{-1}) = u(u(g)\cdot u(g)^{-1}xu(g)x^{-1}\cdot x\xi x^{-1}\xi^{-1}) = u(g) (\mathrm{mod}\, \mathrm{G}\,(r+1))$. Hence $u(g) = u(xgx^{-1})$. It is also clear that since $|\xi| \leqslant p^{-r}$ we have $|z(xgx^{-1})| \leqslant p^{-r}$ and $|v(xgx^{-1})| \leqslant p^{-r}$ the inequality being strict if $|u(g)| > \max(|z(g)|, |v(g)|)$. Hence the first assertion. To prove the second assertion, we argue as follows: $xgx^{-1} = x(u(g)\cdot z(g))x^{-1}\cdot xv(g)x^{-1}v(g)^{-1}$. Since $xu(g)z(g)x^{-1}$ is in P, $v(xgx^{-1}) = v(xv(g)x^{-1}v(g)^{-1})\cdot v(g)$; as $|xv(g)x^{-1}v(g)^{-1}| < |v(g)|$ we conclude that $|v(xgx^{-1})| = |v(g)|$. If $|z(g)| \geqslant |v(g)|$, $|xv(g)x^{-1}v(g)^{-1}| < |z(g)|$ while $z(x\cdot u(g)\, z(g)x^{-1}) = z(g)$ leading to $|z(xgx^{-1})| = |z(g)|$. The last assertion is proved along similar lines.

2.5 The group G is a Lie subgroup of $GL(\mathfrak{g})$ (based on k). Consequently we can find an integer $r > 0$, a neighbourhood W of 0 in $\mathfrak{g}$ and an analytic diffeomorphism $e: W \to \mathrm{G}\,(r)$- with the following properties:

(i) Treating $\mathfrak{g}$ and G as subsets of End $\mathfrak{g}$ and $GL(g)$ respectively, e may be considerd as a map of W in End $(\mathfrak{g})$. The Taylor series of e converges in W and has the following form:

$$e(X) = 1 + X + \sum_{m \geqslant 1} e_m(X), \qquad (*)$$

where e_m for an integer $m > 2$ is a End $\mathfrak{g}$-valued homogeneous polynomial on $\mathfrak{g}$ of degree m.
(ii) e maps $W \cap u'$ (resp. $W \cap \mathfrak{u}$, $W \cap \mathfrak{z}$, $W \cap \mathfrak{v}$ and $W \cap \mathfrak{v}'$) analytically isomorphically onto $\mathrm{G}\,(r) \cap U'$ (resp. $\mathrm{G}\,(r) \cap U$, $\mathrm{G}\,(r) \cap Z$, $\mathrm{G}\,(r) \cap V$ and $\mathrm{G}\,(r) \cap V'$)
(iii) Let $C: W \times W \to \mathfrak{g}$ be the (analytic) map
$C(X, Y) = e^{-1}(e(X)\cdot e(Y)\cdot e(X)^{-1}\cdot e(Y)^{-1})$, $X, Y \in W$. Then c admits a convergent Taylor expansion in $W \times W$ of the form

$$C(X, Y) = [X, Y] + \sum_{r > 0, s > 0, r+s > 2} C_{rs}(X, Y), \qquad (**)$$

where for integers $r, s > 0$, $C_{rs}: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is $\mathfrak{g}$-valued bihomogeneous polynomial on $\mathfrak{g} \times \mathfrak{g}$ of bidegree r, s.

Lemma 2.6. *The Lie bracket operation on* $\mathfrak{g}$ *has the following properties*:

(i) *There is a constant* $A > 0$ *such that for any* $X \in \mathfrak{z}$ *we can find* $Y \in \mathfrak{u} \cap W$ *such that* $\|X\| \leqslant A \,\|[X, Y]\|$
(ii) *There is a constant* $B > 0$ *such that for any* $X \in \mathfrak{u}$ *we can find* $Y \in \mathfrak{u} \cap W$ *such that* $\|\bar{X}\| \leqslant B\, \|[X, Y]\|$ *where* $\bar{X} \in \mathfrak{u}/\mathfrak{u}'$ *is the image of* X *in* $\mathfrak{u}/\mathfrak{u}'$ *under the natural map* $\omega: \mathfrak{u} \to \mathfrak{u}/\mathfrak{u}'$ *and* $\mathfrak{u}/\mathfrak{u}'$ *is identified with* $\mathfrak{g}^{\alpha}$ *through the isomorphism* $\omega|_{\mathfrak{g}^{\alpha}}: \mathfrak{g}^{\alpha} \xrightarrow{\sim} \mathfrak{u}/\mathfrak{u}'$ (*if* $2\alpha \notin \Phi^+$ we set $\mathfrak{u}' = 0$).
(iii) *If* $2\alpha \in \Phi^+$ *there is a constant* $C > 0$ *such that for any* $X \in \mathfrak{u}'$ *and any* $Y \in \mathfrak{v}$ *we have* $\|X\| \cdot \|Y\| \leqslant C\, \|[X, Y]\|$.

The lemma is an easy consequence of the known facts about the structure of k-rank 1 groups: the first assertion is just a reformulation of the fact that $\hat{\rho}^+$ is faithful; the second is a consequence of the fact that the Lie bracket in $\mathfrak{u}$ has the property that, given any $X \in \mathfrak{u}$ with $\bar{X} \neq 0$, there exists $Y \in \mathfrak{u}$ with $[X, Y] \neq 0$; the last assertion is readily deduced from the fact that $\mathbf{G}'$ contains a k-subgroup $\mathbf{H}$ k-isomorphic to $SL(2)$ containing $\mathbf{T}$ and such that $X \in \mathfrak{h}$ (the Lie algebra of $\mathbf{H}$).

Lemma 2.7. *Let $\theta: \mathbf{U}/\mathbf{U}' \to \mathbf{U}$ be the section to $\omega: \mathbf{U} \to \mathbf{U}/\mathbf{U}'$ defined in* 2.1 *and (as in* 2.1*) let $\sigma = \theta \cdot \omega$ and $\tau: \mathbf{U} \to \mathbf{U}'$ be defined by $x = \sigma(x) \cdot \tau(x)$ for all $x \in \mathbf{U}$. There is an integer $d > 0$ such that we have the following inequalities: $p^d \| \omega(x) \| \geqslant |\sigma(x)| \geqslant p^{-d} \| \omega(x) \|$ and $p^d |x| \geqslant \max(|\sigma(x)|, |\tau(x)|) \geqslant p^{-d}|x|$ for all $x \in \mathrm{G}(1) \cap U$. Also if we set $\tau(xy) = \tau(x) \cdot \tau(y) \cdot \psi(x, y)$ we have for $x, y \in \mathrm{G}(1) \cap U$, $|\psi(x, y)| \leqslant p^d |x| |y|$.*

Proof. Since $\sigma(1) = 1$ and $\sigma(x) = \sigma(\omega(x))$ is analytic the first inequality is immediate from the Taylor expansion of σ for x sufficiently close to 1. Since $\mathrm{G}(1) \cap U$ is compct and one has $|x| > \varepsilon > 0$ for all x outside a neighbourhood of 1, the inequality extends to all of $\mathrm{G}(1) \cap U$. For reasons similar to those given above, we need only prove the second inequality for x and y close to 1. Now $\psi(x, y) = \tau(xy)\tau(y)^{-1} \quad \tau(x)^{-1} = xy\sigma(xy)^{-1}\sigma(y)y^{-1}\sigma(x)x^{-1} = \sigma(xy)^{-1}xy \cdot y^{-1}\sigma(y)\sigma(x)x^{-1} = \sigma(xy)^{-1}x \cdot x^{-1}\sigma(x) \cdot (y) = \sigma(xy)^{-1}\sigma(x)\sigma(y)$. Thus $\psi(1, y) = \psi(x, 1) = 1$ for all $x, y \in U$. Taylor expansion of ψ near $(1, 1)$ now gives the desired result.

Lemma 2.8. *There exist positive integers a, b, c such that the following hold.*

(i) *Given $x \in \mathrm{G}(1) \cap Z$, we can find $y \in \mathrm{G}(1) \cap U$ such that*

$$|xyx^{-1}y^{-1}| \geqslant p^{-a}|x|.$$

(ii) *Given $x \in \mathrm{G}(1) \cap U$ we can find $y \in \mathrm{G}(1) \cap U$ such that*

$$|\tau(yxy^{-1})| \geqslant p^{-b}|x| (= p^{-b}|yxy^{-1}|).$$

(iii) *If $2\alpha \in \Phi$ there is an integer $c > 0$ such that for $x \in \mathrm{G}(1) \cap U'$ and $y \in \mathrm{G}(1) \cap V$,*

$$\min(|u(xyx^{-1}y^{-1})|, \max(|z(xyx^{-1}y^{-1})|, |\sigma(u(xyx^{-1}y^{-1}))|)) > p^{-c}|x||y|.$$

Proof. This lemma follows immediately from Lemmas 2.6 and 2.7 and the Taylor expansion (**) of 1.5 for the commutator map. Note that $\tau(yxy^{-1}) = \tau(yxy^{-1}x^{-1}x) = (yxy^{-1}x^{-1})\tau(x)$ for $x, y \in U$ so that in proving (ii) we have only to choose y so that $\max(|yxy^{-1}x^{-1}|, |\tau(x)|) \geqslant p^{-b}|x|$ for a preassigned integer $b \geqslant 0$. From Lemma 2.6 and 2.7 and the Taylor expansion of C we see that we can find $y \in \mathrm{G}(1) \cap U$ such that $|yxy^{-1}x^{-1}| \geqslant p^{-b}|\sigma(x)|$ for a suitable integer $b > 0$ (independent of x). If $|\tau(x)| \geqslant p^{-b}|x|$ we can take $y = 1$; if not $|\sigma(x)| = |x|$ and we can take y to satisfy the inequality above and then $|yxy^{-1}x^{-1}| > |\tau(x)|$ so that $|\tau(yxy^{-1})| = |yxy^{-1}x^{-1}|$. (i) and (iii) are straightforward consequences of the two lemmas and the Taylor expansion for C cited above and the fact that satisfies (ii) of 2.5 above.

2.9 For the concise formulation of later results we now introduce further notations and definitions. We say that an element $x \in \mathrm{G}(1)$ is *P-adapted* if we have

$$|v(x)| \leqslant \max(|u(x)|, |z(x)|) (= |x|). \tag{$*$}$$

It is *U-adapted* if we have

$$|u(x)| \geqslant \max(|z(x)|, |v(x)|). \qquad (**)$$

Evidently if x is *U-adapted* it is affortiori *P-adapted.* The set $E = \{x \in \mathrm{G}(1) | x$ is *P*-adapated$\}$ can also be describe as follows: Let $h \in T$ be any element with $|\alpha(h)| > 1$; then $E = \{x \in \mathrm{G}(1) \,|\, |hxh^{-1}| \geqslant |x|\}$. Note that for any $x \in \mathrm{G}(1)$ either $x \in E$ or $v(x) \in E$. Let E^* be the set of U-adapted elements in $G(1)$. Then one has for $h \in T$ as above and $x \in E^*$, $|hxh^{-1}| \geqslant p|x|$. Finally an element $x \in \mathrm{G}(1)$ is *special* if every element of $\mathrm{G}(1)$ centralising x belongs to E. We denote by S the set of all special elements in $\mathrm{G}(1)$.

If $2\alpha \in \Phi$ let c, d be as in lemmas 2.8 and 2.7. When $2\alpha \notin \Phi$ we set $c = 2d$ and define d as follows: the group G is the adjoint group of either $SL_{2,D}$ D a central division algebra over k or $U(h)$ the unitary group of a non-degenerate, isotropic antihermitian form over a quaternion division algebra. Let $\tilde{G} = GL(2, D)$ in the former case and $\tilde{G} =$ the group of similitudes of h in the latter case: note that $\tilde{G} \subset GL(2, D)$ in both cases with D a suitable division algebra. The natural map $\tilde{G} \xrightarrow{q} G$ is a surjection of maximal rank. Moreover we may assume q so chosen that the group $\tilde{U}$ (resp. $\tilde{V}$) of upper triangular (resp. of lower triangular) matrices in $\tilde{G}$ maps onto U (resp. V) isomorphically while $q|_{\tilde{D}(=\text{diagonal matrices in } \tilde{G})}$ maps $\tilde{D}$ onto Z and is of maximal rank. For $g \in \tilde{G}$, $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$ set $|g| = \max_{1 \leqslant i,j \leqslant 2}(|g_{ij} - \delta_{ij}|)$ and $|g|_0 = \mathrm{Inf}_{x \in k^*}|gx|$; then there is an integer $d > 0$ such that for $g \in \tilde{G}$ with $q(g) \in \mathrm{G}(1)$,

$$p^d|g|_0 \geqslant \max(|u(g)|, |z(g)|, |v(g)|) \geqslant p^{-d}|g|_0.$$

With the above definitions of c and d we define a unipotent element $x \in \mathrm{G}(1)$ to be *hyperspecial* if the following inequality holds.

$$|\tau(u(x))| > p^{c+2d} \max(|\sigma(u(x))|, |z(x)|, |v(x)|). \qquad (*)$$

This the terminology is justified by

Lemma 2.10. *Any hyperspecial element is special.*

Proof. We deal with the case $2\alpha \in \Phi$ first. Let x be a hyperspecial element and $y \in \mathrm{G}(1)$ an element commuting with x. Let $\tau(u(x)) = \rho$, $\sigma(u(x)) \cdot z(x) \cdot v(x) = \xi$, $u(y) \cdot z(y) = \eta$ and $v(y) = \alpha$. Assume that $|\alpha| > \max(|u(y)|, |z(y)|) = |\eta|$: we show that this leads to a contradiction. For $g, h \in G$, we denote $ghg^{-1}h^{-1}$ by $[g, h]$. With this notation we have

$$[x, y] = \rho[\xi, \eta]^{-1} \cdot [\rho, \eta] \cdot \eta\rho[\xi, \alpha]^{-1}\eta^{-1} \cdot \eta[\rho, \alpha]\eta^{-1}.$$

Now let $|\rho| = p^{-l}$ and $|\alpha| = p^{-m}$ and set $n = l + m + c + 2d + 1$. Since $|\alpha| > |\eta|$ and $|\xi| < p^{-(c+2d)}|\rho|$ we see that we have

$$1 = [x, y] = [\rho, \eta] \cdot \eta[\rho, \alpha]\eta^{-1} \,(\mathrm{mod}\, \mathrm{G}(n)).$$

Now as $\rho \in U'$ and $\eta \in P$, $[\rho, \eta] \in U'$. Thus we see that $u([x, y]) = [\rho, \eta]\, u([\eta, \alpha]\eta)^{-1}$ $(\mathrm{mod}\, \mathrm{G}(n))$, $z([x, y]) = z(\eta[\rho, \alpha]^{-1}$ $(\max \mathrm{G}(n))$ and $v([x, y]) = v(\eta[\rho, \alpha]\eta^{-1})$ $(\mathrm{mod}\, \mathrm{G}(n))$. Now if $v([\rho, \alpha]) \notin \mathrm{G}(n)$, $v(\eta[\rho, \alpha]\eta^{-1}) \notin \mathrm{G}(n)$ (Lemma 1.4) so that $v([x, y]) \notin \mathrm{G}(n)$, a contradiction. Thus we may assume that $v([\rho, \alpha]) \in \mathrm{G}(n)$. Again appealing to Lemma 1.4 we see that $z([x, y]) = z([\rho, \alpha])$ $(\mathrm{mod}\, \mathrm{G}(n))$. This means that we have

necessarily $z([\rho,\alpha])\in G\,(n)$. In other words $|z([\rho,\alpha])|<p^{l+m+c+2d}$. We also have

$$1=[x,y]=[\rho,\eta]\cdot u(\eta[\rho,\alpha]^{-1}) \bmod G\,(n)$$

and since $\eta[\rho,\alpha]\eta^{-1}=u([\rho,\alpha])\eta^{-1}\,(\bmod G\,(n))$ we have in fact

$$1=[x,y]=[\rho,\eta]\cdot\eta\cdot u([\rho,\alpha])\eta^{-1}\,(\bmod G\,(n)).$$

Now $|u([\rho,\alpha])|=|\eta u([\rho,\alpha])\eta^{-1}|$. It follows that $|\sigma(\eta\cdot u([\rho,\alpha])\cdot\eta^{-1})|\geqslant p^{-d}|\eta u([\rho,\alpha])\eta^{-1}|=p^{-d}|u([\rho,\alpha])|$. It follows from Lemma 1.8 – since $|z([\rho,\alpha])|\leqslant p^{-n}$ – that $|u([\rho,\alpha])|\geqslant p^{-l-m-c}$ so that $\sigma(\eta\cdot u([\rho,\alpha])\eta^{-1})\geqslant p^{-l-m-c-d}$. On the other hand we have $[\rho,\eta]\cdot(\eta u([\rho,\alpha])\eta^{-1})\in G(n)$ and hence by Lemma 1.7 once again (since $\sigma([\rho,\eta]\cdot\eta u([\rho,\alpha])\eta^{-1})=\sigma(\eta u([\rho,\alpha])\eta^{-1}))$ we see that we have $|\sigma(\eta u[\rho,\alpha]\eta^{-1})|\leqslant p^{d}\cdot p^{-n}<p^{-l-m-c-d}$, a contradiction. This proves the lemma in the case when $2\alpha\in\Phi$. Assume now that $2\alpha\notin\Phi$ and $\tilde{G}$ be the group introduced in 2.9. Let x be hyperspecial so that

$$|u(x)|=|\tau(u(x))|>p^{4d}\max(|z(x)|,|v(x)|).$$

Let $\tilde{z}(x)\in\tilde{D}$ be any lift of $z(x)$; set $\tilde{x}=\tilde{u}(x)\,\tilde{z}(x)\,\tilde{v}(x)$ where $\tilde{u}(x)$ (resp. $\tilde{v}(x)$) is the unique lift of $u(x)$ (resp. $v(x)$) to $\tilde{U}$ (resp. $\tilde{V}$). In view of the definition of d, one has a lift $\tilde{z}(x)$ of $z(x)$ to D such that

$$|\tilde{u}(x)|>p^{2d}\max(|\tilde{z}(x)|,|\tilde{v}(x)|).$$

Suppose now that $y\in Z(x)$ ($=$centraliser of x) and $\tilde{y}$ is any lift of y in $\tilde{G}$; then the map $x^r\to x^{-r}\tilde{y}\tilde{x}^{-r}\tilde{y}^{-1}=\psi_y(x^r)$ depends only on x^r and y (and not on the lifts $\tilde{x}$ and $\tilde{y}$ chosen) and is a homomorphism of the cyclic p-group generated by x into k^* (since $y\in Z(x)$). Since char $k=p$, ψ_y is trivial i.e. $\tilde{x}$ and $\tilde{y}$ commute. If we set $\tilde{y}=\tilde{u}(y)\cdot\tilde{z}(y)\,\tilde{u}(y)$ with $\tilde{u}(y)$ (resp. $\tilde{z}(y)$, resp. $\tilde{v}(y)$) in $\tilde{U}$ (resp. $\tilde{D}$, resp. $\tilde{V}$), a simple matrix computation shows that

$$|\tilde{v}(y)|\leqslant p^{-2d}\max(|\tilde{z}|y)|,|\tilde{u}(y)|).$$

As this holds for any lift $\tilde{y}$ of y to G we conclude that

$$|\tilde{v}(y)|_0\leqslant p^{-2d}\max(|\tilde{z}(y)|_0,|u(y)|_0).$$

Once again from the definition of d we have

$$|v(y)|\leqslant\max(|z(y)|,|u(y)|).$$

Hence the lemma.

PROPOSITION 2.11

Let $N=a+2b+c+6d$ where a,b,c are as in Lemma 2.8 and d is as in Lemma 2.7 and $t\in T$ the unique element such that $\alpha(t)=\pi^{-N}$. Let $C\subset P$ be the compact set $t\cdot(G(1)\cap U)\cdot t\cdot(G(1)\cap U).\,t$. Then given any $x\in E$ with $|x|<p^{-6N}$ there exists $g\in C$ such that the following holds

$|gyg^{-1}|\geqslant|y|$ for all $y\in E$

$|gyg^{-1}|\geqslant\min(p^{3N}|y|,p)$ for all $y\in E^*$

(iii) $|gyg^{-1}|, |g^{-1}yg| \leqslant \min(p^{6N}|y|, p)$ for all $y \in E$
(iv) $|gxg^{-1}| > p^N|x|$
(v) gxg^{-1} is hyperspecial.

Proof. Observe that for any $h \in T$ with $|\alpha(h)| > 1$, one has $|hyh^{-1}| \geqslant |y$ $y \in E$ while $|hyh^{-1}| \geqslant \min(p, |\alpha(h)| \cdot |y|)$ if $y \in E^*$. It is also clear that $|hyh^{-1}| \leqslant \min(p, |\alpha(h)^2||y|)$. Since $|gyg^{-1}| = |y|$ for all $g \in G$ while E and E^* are stable under linear conjugation by $U \cap \mathrm{G}(1)$, we see that the first three inequalities hold for *any* $g \in C$. Thus we have only to find $g \in C$ to satisfy (iv) and (v). In the sequel we will define the elements x_i, $1 \leqslant i \leqslant 4$ and x' in G (1) and we will set $u(x_1) = u_i$, $z(x_i) = z_i$ $v(x_i) = v_i$, $u(x') = u'$, $z(x') = z'$, $v(x') = v'$, $u(x) = u$, $z(x) = z$ and $v(x) = v$. We define $x_1 = txt^{-1}$. Then $u_1 = tut^{-1}$, $z_1 = tzt^{-1} = z$ and $v_1 = tvt^{-1}$ so that $\max(|z_1|, |u_1|) \geqslant p^N|v_1|$. In particular $|x_1| = \max(|z_1|, |u_1|) \geqslant |x|$. Now choose an element $\xi \in \mathrm{G}(1) \cap U$ as follows: if $|u_1| \geqslant p^{-a} \cdot |z_1|$, $\xi = 1$; if not ξ is such that $|\xi z_1 \xi^{-1} z_1^{-1}| \geqslant p^{-a}|z_1|$. (Lemma 2.8). Let $x_2 = \xi x_1 \xi^{-1}$; then $|v_2| = |v_1|$, $|z_2| = |z_1|$ while $|u_2| \geqslant p^{-a}|z_2|$ and also $|x_2| = |x_1| = \max(|u_2|, |z_2|) \geqslant p^N|v_2|$. Next let $x_3 = tx_2t^{-1}$; then we have $u_3 = tu_2t^{-1}$ so that $|u_3| \geqslant p^N|u_2| \geqslant p^{N-a}|z_2| = p^{N-a}|z_3|$ (since $z_3 = tz_2t^{-1} = z_2$) and $|v_3| \leqslant p^{-N}|v_2| = \leqslant p^{-2N}\max(|u_2|, |z_2|) \leqslant \max(p^{-3N}|u_3|, p^{-2N}|z_3|)$ $(\leqslant p^{-3N+a}|u_3|)$. It is also clear now that $|x_3| \geqslant p^{N-a}|x_2|$. We now choose an element $\eta \in \mathrm{G}(1) \cap U$ as follows: if $|\tau(u_3)| \geqslant p^{-b}|u_3|$ we set $\eta = 1$; if not choose η such that $\tau(\eta u_3 \eta^{-1}) \geqslant p^{-b}|u_3|$ (Lemma 2.8). Let $x_4 = \eta x_3 \eta^{-1}$. Then $|x_4| = |x_3|$. We claim that $|\tau(u_4)| \geqslant p^{-b}|u_4| = p^{-b}|u_3|$. When $\eta = 1$, this is evident. Hence we assume that $\eta \neq 1$ i.e. $|\tau(u_3)| < p^{-b}|u_3|$. Since we have $|u_3| > |z_3| > |v_3|$ and $\eta \in \mathrm{G}(1) \cap U$ it follows from Lemma 2.4 that $|u_4| = |u_3|$, $|z_4| = |z_3|$ and $|v_4| = |v_3|$; now $x_4 = \eta \cdot x_3 \eta^{-1} = \eta u_3 \eta^{-1} \cdot \eta z_3 v_3 \eta^{-1}(z_3v_3)^{-1} \cdot z_3v_3$ and as $z_3v_3 \in P^-$, we have $u_4 = \eta u_3 \eta^{-1} \cdot u([\eta, z_3v_3]) = [\eta, u_3] \cdot u_3 \cdot u([\eta, z_3v_3])$. Thus we have

$$\tau(u_4) = [\eta, u_3] \cdot \tau(u_3 \cdot u([\eta, z_3v_3]))$$

Now $|\tau(u_3)| < p^{-b}|u_3|$ while $|[\eta, z_3v_3]| < p^{-N+a}|u_3|$ so that $|u([\eta, z_3v_3])| < p^{-N+a}|u_3|$ as well. By Lemma 2.7, one has $\tau(u_3 \cdot u([\eta, z_3v_3])) = \tau(u_3) \cdot \tau(u([\eta, z_3v_3]))\psi(u_3, u([\eta, z_3v_3])$. With $|\psi(u_3, u([\eta, z_3u_3]))| \leqslant p^d$ $|u_3| \cdot u([\eta, z_3v_3] < p^{-N+a+d}|u_3|$. On the other hand $|\tau(u_3)| < p^{-b}|u_3|$ and $\tau(u([\eta, z_3v_3])) \leqslant p^d \cdot |u([\eta, z_3v_3]) > p^{-N+a+d}|u_3|$. Since $N - a - d > b$ and $|[\eta, u_3]| \geqslant p^{-b}$ $|u_3|$, we can now conclude that $\tau(u_4) \geqslant p^{-b}$ $|u_3| = p^{-b}|u_4|$. We have also $|\sigma(u_4)| \leqslant p^d|u_4|$. Finally let $x' = tx_4t^{-1}$. Then we have $u' = tu_4t^{-1}$ $z' = tz_4t^{-1} = z_4$ and $v' = tv_4t^{-1}$. Thus $|v'| \leqslant p^{-N}$ $|v_4| < p^{-N}|z_4| = p^{-N}|z'|$. Next observe that σ and τ are compatible with the action of T so that $\sigma(u') = t\sigma(u_4)t^{-1}$ and $\tau(u') = t$ $\tau(u_4)t^{-1}$. It follows that $|\tau(u')| = |\alpha(t)|^2 \cdot |\tau(u_4)| = p^{2N}|\tau(u_4)| \geqslant p^{2N-b}$ $|u_4|$. On the other hand we have $|\sigma(u')| \leqslant p^d \| \omega\sigma(u') \| = p^{d+N} \| \omega\sigma(u_4) \| \leqslant p^{2d+N}|\sigma(u_4)| \leqslant p^{3d+N}$ $|u_4|$. Thus $|\sigma(u')| \leqslant p^{3d+N}|u_4| \leqslant p^{3d+N-2N+b}|\tau(u')|$; and $N - b - 3d \geqslant c + 2d$. Also, $|u_4| = |u_3| \geqslant p^{N-1}$ $|z_3| = p^{N-a}|z_4|$ leading to $|z'| = |z_4| \leqslant p^{-N+a}|u_4| \leqslant p^{-N+a}$ p^{-2N+b} $|u'|$; and $-3N + a + b > c + 2d$. Thus x' is hyperspecial. If we set $g = t\eta t\xi t$, $g \in C$ and $x' = gxg^{-1}$. From our definitions it is easy to see that $|x'| > p^N|x|$. Thus the proposition is proved.

Lemma 2.12. *There is a compact set $K \subset G$ such that $K \supset$ G and for any $x \in G$ contained in the unipotent radical of a k-parabolic subgroup of* **G**, *we can find $g \in K$ such that $gxg^{-1} \in U$. Also if $x \in \mathrm{G}(1)$, we can find $g \in K$ such that $gxg^{-1} \in E$.*

Proof. Since G/P is compact and any two minimal k parabolic subgroups of G are conjugate by an element of G and for any $x \in \mathrm{G}$ (1) either $x \in E$ or $vxv^{-1} \in E$, the lemma follows.

3. Bound on "covolumes" and existence of good unipotents

3.1 *Notation*

We now deviate somewhat from the notation of §2. We will denote by G a direct product of groups of the form $G_i = \mathbf{G}_i(k_i)$ $1 \leqslant i \leqslant q$ where for each i, k_i is a local field of characteristic $p > 0$ (independent of i) and $\mathbf{G}_i$ is a connected absolutely almost simple k-algebraic group of adjoint type of k-rank 1. We fix in each $\mathbf{G}_i$ a split torus $\mathbf{T}_i$ and let $\mathbf{U}_i$, $\mathbf{V}_i$ denote the two maximal k_i-split unipotent subgroups normalised by $\mathbf{T}_i$. Also $\mathbf{Z}_i$ and $\mathbf{N}_i$ denote respectively the centraliser and normaliser of $\mathbf{T}_i$ in $\mathbf{G}_i$ and set $\mathbf{\Omega}_i = \mathbf{U}_i\mathbf{Z}_i\mathbf{V}_i$. Applying the results of §2 to G_i we obtain a family of compact open subgroups $\mathrm{G}_i(r)$, r an integer $\geqslant 0$, in G_i such that for $r \geqslant 1$, $\mathrm{G}_i(r)$ is a pro-p group, $\mathrm{G}_i(r) \supset \mathrm{G}_i(r+1)$ and $[\mathrm{G}_i(r), \mathrm{G}_i(s)]$ is contained in $\mathrm{G}_i(r+s)$. This family of groups define a metric d_i on G_i if we set

$$d_i(g,h) = \begin{cases} p & \text{if } g^{-1}h \notin \mathrm{G}_i = \mathrm{G}_i(0) \\ \inf\{p^{-r} \mid g^{-1}h \in \mathrm{G}_i(r)\} & \text{if } g^{-1}h \in \mathrm{G}_i \end{cases}$$

and let $|g|_i = d_i(1,g)$ for all $g \in G_i$. We assume $\mathrm{G}_i(1)$ to be so chosen that it is contained in $\Omega_i = \mathbf{\Omega}_i(k_i)$ and denote by E_i (resp. E_i^*) the set $x \in \mathrm{G}_i(1)$, $x = uzv$, $u \in U_i = \mathbf{U}_i(k_i)$ $z \in Z_i = \mathbf{Z}_i(k_i)$ and $v \in V_i = \mathbf{V}_i(k_i)$ with $|v|_i \leqslant \max(|u|_i, |z|_i)$ (resp. $|u_i| > \max(|z_i|, |v_i|)\}$. We also denote by S_i the set $\{x \in E_i^* \mid$ centraliser of x in G_i is contained in $E_i\}$ and call the elements of S_i special elements in G_i. Finally let U (resp. $T, V, Z, N, \Omega, \mathrm{G}_i(r), E, E^*, S$) denote the product of the U_i (resp. $T_i, V_i, Z_i, N_i, \Omega_i, \mathrm{G}_i(r), E_i, E_i^*, S_i$) considered as a subset of G. Also we set for $g, h \in G$, $d(g,h) = \max\{d_i(g_i, h_i) \mid 1 \leqslant i \leqslant q\}$ where $g_i h_i$ are the components of g, h in the factor G_i and denote by $|g|$ the distance $d(1,g)$ of g from 1 in the metric d. With these new notations we have the following result which is essentially a reformulation of Proposition 2.11 and Lemma 2.12 in the new notation.

PROPOSITION 3.2

There is an integer $N > 0$ and an element $t \in T$ such that the following holds. Let $C = t(\mathrm{G}(1) \cap U)t(\mathrm{G}(1) \cap U) \cdot t$. Then E, E^* and S are stable under inner conjugation by elements of C. Further we have

(i) $|gyg^{-1}| \geqslant y$ *for all* $y \in E$ *and* $g \in C$

(ii) $|gyg^{-1}| \geqslant \min(p^{3N}|y|, p)$ *for all* $y \in E^*$ *and* $g \in C$

(iii) $|gyg^{-1}| \leqslant \min(p^{6N}|y|, p)$ *for all* $y \in \mathrm{G}(1)$ *and* $g \in C$ *or* $g^{-1} \in C$.

Further given any $x \in E$ we can choose a $g \in C$ such that

(iv) $|gxg^{-1}| \geqslant p^N|x|$

(v) $gxg^{-1} \in S$.

There is a compact subset K of G containing 1 with the following property. Given any $x \in G$ whose components x_i are contained in the unipotent radicals of k_i-parabolic

subgroups of the $\mathbf{G}_i$, $1 \leqslant i \leqslant q$, there is a $g \in K$ such that $gxg^{-1} \in U$. Given any $x \in \mathrm{G}$ (1) there is a $g \in K$ such that $gxg^{-1} \in E$. (and hence) an element $g' \in C \cdot K$ such that $g'xg'^{-1}$ is special. Also for $g \in K$ and $x \in \mathrm{G}$ (1), we have $|gxg^{-1}| < p^N|x|$.

DEFINITIONS 3.3

A subgroup U' of G is a *horocycle* if it is conjugate to U in G. A subgroup P' of G is a *parabolic subgroup* if it is conjugate to $P(=$normaliser of U in $G)$. Such a P' contains a unique conjugate U' of U. The subgroup U' of P' is normal in P' and will be referred to as the *nilradical* of P'. An element $g \in G$ is *unipotent* if $g^{p^r} = 1$ for some integer $r \geqslant 0$. A subgroup of G is unipotent if every element in it is unipotent. A unipotent element is *good* if it has a conjugate in U (equivalently if it belongs to a horocycle).

Lemma 3.4. *Any unipotent element x in $[G, G]$ is good.*

Proof. It is clear from the definition that $x \in G$ is a good unipotent if and only if every i, $1 \leqslant i \leqslant k$, the component x_i of x in G_i is the unipotent radical of a k_i-parabolic subgroup of $\mathbf{G}_i$ i.e. x_i is good in the sense of Borel-Tits [3]. Let $\tilde{\mathbf{G}}_i$ be the simply connected covering of $\mathbf{G}_i$. Then according to Borel-Tits [3] every unipotent in $\tilde{\mathbf{G}}_i(k_i)$ is good. The lemma follows from the fact that the cokernel of the natural map $\tilde{\mathbf{G}}_i(k_i) \to G_i$ is *abelian.*

Lemma 3.5. *Let $x \in G$ be a good unipotent element such that $x_i \neq 1$ for all i, $1 \leqslant i \leqslant q$ where x_i is the component of x in G_i. Then x is contained in a unique parabolic subgroup $P(x)$ of G. Further x belongs to the nilradical $U(x)$ of $P(x)$. If $g \in G$ is such that x and gxg^{-1} generate a unipotent subgroup of g, then $g \in P(x)$.*

Proof. One reduces the problem to the case when $q = 1$ by looking at the projections into the various factors. Clearly, then assuming that $G = \mathbf{G}(k)$, $\mathbf{G}$ a connected absolutely simple adjoint group over k of k-rank 1, it suffices to prove the following assertion: if $x \in P$ is good and $y^{-1}xy \in P$, then $y \in P$. To see this we make use of Bruhat-decomposition. If $y \notin P$, $y = unu'$ where u, $u' \in U$ and $n \in N \backslash Z$ uniquely. Let $x' = u^{-1}xu$. Then since $u'Pu'^{-1} \subset P$ we conclude that $n^{-1}x'n \in P$. But for any $n \in N - Z$, $n^{-1}Un \subset V$ and $x' \neq 1$ belongs to U, a contradiction. The uniqueness of the parabolic subgroup containing x is thus proved. We denote this subgroup by $P(x)$. Suppose now that $x \in U$ and P' is a parabolic subgroup containing x. Since $P' = g^{-1}Pg$ for some $g \in G$, we have $gxg^{-1} \in P$ so that $g \in P$ and hence $gxg^{-1} \in U =$ nilradical of P'. Thus if a good unipotent belongs to a parabolic subgroup it belongs to its nilradical and hence the second statement. Now let $y = gxg^{-1}$ and Ψ the (unipotent) subgroup generated by x and y. Let $z \in \Psi$ be a nontrivial good unipotent centralising x and y: if Ψ is abelian then we can take $y = z$; if not $[\Psi, \Psi]$ consists entirely of good unipotents and we can take for z any nontrivial element in the last term of the descending central series of (the nilpotent group). Thus it suffices to prove the following: if g and h are commuting nontrivial good unipotents, then $P(g) = P(h)$. Now $gP(h)g^{-1} = P(ghg^{-1}) = P(h)$ so that $g \in P(h)$ leading to $P(g) = P(h)$.

The following well-known lemma is recorded for future use.

Lemma 3.6. *For any parabolic subgroup P' G, G/P' is compact.*

Lemma 3.7. *Let $\Gamma \subset G$ be a discrete subgroup. Then we have*

(i) $\Gamma \cap \mathrm{G}\ (1)$ *is a finite unipotent group*
(ii) *If* $\Gamma \cap \mathrm{G}\ (1)$ *contains a good unipotent* x *all of whose components in the* G_i *are non-trivial, then* $\Gamma \cap \mathrm{G}\ (1) \subset P(x)$
(iii) *If* $\Gamma \cap \mathrm{G}$ *does not contain a non-trivial good unipotent it is abelian.*

Proof. G(1) is a pro-p group. The first assertion follows from this. If $x, g \in \Gamma \cap \mathrm{G}\ (1)$ and x is good unipotent x and gxg^{-1} generate a unipotent group. By Lemma 3.5 $g \in P(x)$ if all the components of x are non-trivial; hence the second assertion. The third follows from Lemma 3.4.

DEFINITIONS 3.8

A discrete subgroup $\Gamma \subset G$ is *irreducible* if the restriction to Γ of the Cartesian projection of G on G_i is injective for all i with $1 \leqslant i \leqslant q$. An *irreducible* discrete subgroup $\Gamma \subset G$ is in *good position* if either $\Gamma \cap \mathrm{G}\ (1) \cap U^+$ is non-trivial or $\Gamma \cap \mathrm{G}\ (1)$ contains an element of S in its centre and $\Gamma \cap \mathrm{G}\ (6N+1)$ does not contain any nontrivial good unipotents.

Lemma 3.9. *If* Γ *is irreducible and in good position,* $\Gamma \cap \mathrm{G}\ (1) \subset E$.

Proof. If $1 \neq x \in \Gamma \cap \mathrm{G}\ (1) \cap U$ one has, $\Gamma \cap \mathrm{G}\ (1) \subset P(x) = P \subset E$ (Lemma 3.7). If on the other hand $\Gamma \cap \mathrm{G}\ (1) \cap U = \{1\}$, by definition $\Gamma \cap \mathrm{G}\ (1)$ contains an element x of S in its centre. Thus $\Gamma \subset \mathrm{G}\ (1) \subset$ centraliser x; and centraliser $x \subset E$.

Lemma 3.10. *Let* C *and* K *be compact sets as in Proposition* 3.2. *Then given any irreducible discrete subgroup* $\Gamma \subset G$ *with* $\Gamma \cap \mathrm{G}(8N) \neq 1$ *we can find* $g \in C'K \cup CK \cup K.C.K$ *such that* $g\Gamma g^{-1}$ *is in good position.*

Proof. We assume that Γ is not in good position. Suppose now that $\Gamma \cap \mathrm{G}\ (N+1)$ contains a nontrivial good unipotent x. Choose $g \in K$ such that $gxg^{-1} \in U$; since $|gxg^{-1}| \leqslant p^N |x|$, $|gxg^{-1}| \leqslant p^{-1}$ so that $g\Gamma g^{-1}$ is in good position. Consider now the case when $\Gamma \cap \mathrm{G}\ (N+1)$ does not contain any nontrivial unipotent. In this case pick an element $x \neq 1$ in $\Gamma \cap \mathrm{G}\ (8N)$ and an element $g \in C \cdot K$ such that $x' = gxg^{-1}$ is in S. Let $\Gamma' = g\Gamma g^{-1}$; if $\Gamma' \cap G\ (N+1)$ contains a nontrivial good unipotent y' we can find $h \in K$ such that $hy'h^{-1} \in \Gamma'' \cap \mathrm{G}\ (1) \cap U$ where $\Gamma'' = h\Gamma h^{-1} = hg\Gamma g^{-1}h^{-1}$. Since $hg \in K \cdot CK$ and Γ'' are in good position we need only deal with the case when Γ' is such that $\Gamma' \cap U \cap \mathrm{G}\ (1) = \{1\}$ and $\Gamma' \cap \mathrm{G}(N+1)$ does not contain any nontrivial good unipotents. In this case we claim that Γ' is in good position. To see this, observe that we have only to prove that the element $x'(\varepsilon \Gamma' \cap \mathrm{G}\ (N) \cap S)$ is central in $\Gamma' \cap \mathrm{G}\ (1)$. Now $x' \in \Gamma' \cap \mathrm{G}(N)$ and if $y \in \Gamma' \cap \mathrm{G}\ (1)$, $xyx^{-1}y^{-1}$ is a good unipotent contained in $\Gamma' \cap \mathrm{G}\ (N+1)$ and must hence be trivial. Thus x' is central in $\Gamma' \cap \mathrm{G}\ (1)$.

PROPOSITION 3.11

Let $\Gamma \subset G$ *be an irreducible descrete subgroup in good position with* $\Gamma \cap \mathrm{G}\ (12N) \neq 1$. *Let* $1 \neq x \in \Gamma \cap \mathrm{G}\ (12N)$. *Then there exists* $g \in C$ *such that*

(i) $|gyg^{-1}| \geqslant |y|$ *for all* $y \in \Gamma \cap \mathrm{G}\ (1)$
(ii) $|gyg^{-1}| \geqslant \min(p^{3N}|y|, p)$ *for all* $y \in \Gamma \cap \mathrm{G}\ (1) \cap E^*$
(iii) $|gxg^{-1}| \geqslant p^N |x|$

(iv) gxg^{-1} *is special.*

(v) $g\Gamma g^{-1}$ *is in good position.*

(vi) $d(g\Gamma g^{-1}) < d(\Gamma)$.

(Here for a discrete subgroup $\Psi \subset G$, $d(\Psi) = \sum_{x \in \Psi \cap \mathrm{G}(12N)\setminus\{1\}} -\log x$.

Proof. By Lemma 3.9, $\Gamma \cap \mathrm{G}\ (1) \subset E$. Choose now g as in Proposition 3.11. Then evidently g satisfies all the requirements (i)–(iii) above. We will now show that $g\Gamma g^{-1}$ is in good position as well. Let $x' = gxg^{-1}$. $\Gamma' = g\Gamma g^{-1}$. If $\Gamma' \cap \mathrm{G}(1) \cap U \neq \{1\}$, there is nothing to prove. We assume then that $\Gamma' \cap \mathrm{G}\ (1) \cap U = \{1\}$. Suppose now that $\Gamma \cap \mathrm{G}\ (6N+1)$ contains a nontrivial good unipotent y'; then $y' = gyg^{-1}$ with $y \in \Gamma \cap \mathrm{G}\ (1)$. But in view of (i) we have $y \in \Gamma \cap \mathrm{G}\ (6N+1)$. As Γ is in good position and y is a good unipotent, $y \in U$. Now $g \in C \subset P$ so that $y' = gyg^{-1} \in U$, a contradiction to our assumption that $\Gamma' \cap \mathrm{G}\ (1) \cap U = \{1\}$. We conclude thus that $\Gamma' \cap \mathrm{G}\ (6N+1)$ contains no nontrivial good unipotents. Finally as x is in $\mathrm{G}\ (12N)$, $x' \in \mathrm{G}\ (6N)$ so that for $\xi \in \Gamma \cap \mathrm{G}(1)$ $x'\xi x'\xi^{-1}$ is a good unipotent contained in $\Gamma' \cap \mathrm{G}\ (N+1)$; thus x' is central in $\Gamma' \cap \mathrm{G}(1)$. It follows that Γ' is in good position. To prove (vi) we note that if $y' = gyg^{-1}$, $y \in \Gamma$, belongs to $\mathrm{G}\ (12N)$, $y \in \mathrm{G}\ (6N)$; but then $|y'| \geqslant |y|$ so that $y \in \mathrm{G}\ (12N)$. It is thus clear that we have $F' = \{y \in \Gamma | gyg^{-1} \in \mathrm{G}\ (12N)\} \subset \{y \in \Gamma | y \in \mathrm{G}\ (12N)\} = F$ and that for every $y \in F'$, $|gyg^{-1}| \geqslant |y|$ with strict inequality for at least one element if $F' = F$. It is clear from this that we have $d(g\Gamma g^{-1}) < d(\Gamma)$.

Theorem 3.12. *Given any irreducible discrete subgroup* $\Gamma \subset G$ *there is an element* $g \in G$ *such that* $g\Gamma g^{-1} \cap \mathrm{G}\ (12N) = \{1\}$. *(Consequently) if* μ *is a Haar measure on* G, *for any discrete subgroup* $\Gamma \subset G$, *the volume of* G/Γ *for the measure derived from* μ *is bounded below by a constant* $C > 0$ *depending only* G *(and* μ*).*

Proof. The first assertion is an immediate consequence of the Proposition. For irreducible Γ, the second assertion follows from the first since $\mathrm{G}\ (12N)$ maps injectively into $G/g\Gamma g^{-1}$. For general Γ we need only prove the assertion under the assumption that volume (G/Γ) is finite; and in that case we can decompose G into a direct product so that Γ is contained in a corresponding product of irreducible discrete subgroups in the different factors.

3.13 Proposition 3.11 carries much more information than we have used in the proof of Theorem 3.12. Suppose that C and C' are compact subsets of G as in Lemma 3.10 and Γ_0 is an irreducible *L-subgroup* in G. Then there is a *compact* set $B_0 \subset G$ such that

$$B_0\Gamma_0 \supset \{g \in G | g\Gamma_0 g^{-1} \cap \mathrm{G}(12N) = \{1\}\}.$$

Let $A_0 = \{ghxh^{-1}g^{-1} | x \in \mathrm{G}\ (12N), g^{-1} \in CUC' \text{ and } h \in B_0\}$. Clearly A_0 is compact. Let

$$\Delta = \{\theta \in A \cap \Gamma_0 | \theta \neq 1 \text{ and } g\theta g^{-1} \in \mathrm{G}\ (12N) \text{ for some } g \in G\}.$$

Evidently Δ is finite. Let Δ_0 be the subset of all good unipotents in Δ and Δ_0' its complement in Δ. Now $x \in G$ is a good unipotent if and only if G-orbit of x under inner conjugation contains 1 in its closure. Thus there is an integer $N' > 0$ such that we have for all $\theta \in \Delta_0'$ and $g \in G$,

$$|g\theta g^{-1}| \geqslant p^{-N'}$$

Let $e = |\Delta'_0|$, $f = \max(1, [(N' - 12N)/3] + 1)$ and $N_1 = 12N + 6N(e+f)$. Let $B_1 \subset G$ be a compact set containing B_0 such that $B_1\Gamma_0 \supset \{g \in G | g_0 g^{-1} \cap \mathrm{G}\ (N_1)$ is trivial$\}$. Let $A_1 = \{ghxh^{-1}g^{-1} | x \in \mathrm{G}\ (12N),\ g^{-1} \in C \cup C',\ h \in B_1\}$ and $\Delta_1 = \{\delta \in \Gamma_0 \cap A_1 | \delta$ a nontrivial good unipotent$\}$.

Let $g \in G$ be any element and consider the discrete subgroup $g\Gamma_0 g^{-1}$. Choose an element $h_0 \in C'$ such that $h_0 g \Gamma_0 g^{-1} h_0^{-1} = \Gamma_1$ is in good position. We will define inductively conjugates $\Gamma_2, \Gamma_3 \ldots$ of Γ in G as follows. Assume Γ_i, $1 \leqslant i \leqslant l$ is defined to satisfy the following conditions. There are elements $h_i \in C \cup \{1\}$, $1 \leqslant i < l$ such that

(1°) $\Gamma_{i+1} = h_i \Gamma_i h_i^{-1}$ for $1 \leqslant i < l$

(2°) If $\Gamma_i \cap \mathrm{G}\ (12N) = \{1\}$, $h_i = 1$.

(3°) If $\Gamma_i \cap \mathrm{G}\ (12N) \neq \{1\}$, then (h_i, Γ_i) fulfils all the conditions on (g, Γ) in Proposition 3.2 with the element $x_i \neq 1$ in $\Gamma_i \cap \mathrm{G}\ (12N)$ taking the place of x in Proposition 3.2 chosen in the following fashion: let $b_i = h_i h_{i-1} \cdots h_0 g$ so that $\Gamma_{i+1} = b_i \Gamma_0 b_i^{-1}$; then if $(\Gamma_i \cap \mathrm{G}(12N)) \cap E^*$ contains a Γ_i-conjugate of some element of $b_i \Delta'_0 b_i^{-1}$ we take x_i to be such a conjugate; otherwise x_i is taken to be any nontrivial element of $\Gamma_i \cap G(12N)$.

Now if $\Gamma_l \cap \mathrm{G}\ (12N) = \{1\}$ we set $h_l = 1$; if $\Gamma_l \cap \mathrm{G}\ (12N) \neq \{1\}$, choose $h_l \in C$ so that (h_l, Γ_l) satisfy the conditions on (g, Γ) in Proposition 3.11 taking in place of x an element x_l in $\Gamma_l \cap \mathrm{G}\ (12N)$, $x_l \neq 1$ which is a conjugate in Γ_l of an element $b_l \delta'_0 b_l^{-1}$, $\delta'_0 \in \Delta'_0$, if such a conjugate exists in $\Gamma_l \cap \mathrm{G}\ (12N)$. Note that all the Γ_i constructed are in good position and there is an integer $r > 0$ such that $b_r \Gamma b_r^{-1} \cap \mathrm{G}\ (12N) = \{1\}$ while (if $r > 1$), $b_{r-1}\Gamma b_{r-1}^{-1} \cap \mathrm{G}\ (12N) \neq \{1\}$.

3.14 Let $\Psi_i = \{x \in \Gamma | b_{i-1} x b_{i-1}^{-1} \in \mathrm{G}\ (12N)\}$. Then one has

$$\Psi_{r+1} = \{1\} \subset \Psi_{i+1} \subset \Psi_i \subset \Psi_1$$

for $1 \leqslant i \leqslant r$. Also, clearly for $1 \leqslant i \leqslant r$ we have

$$d(\Gamma_i) = -\sum_{x \in \Psi_i} \log |b_{i-1} x b_{i-1}^{-1}| > 0$$

while $d(\Gamma_{r+1}) = 0$. For $\gamma \in \Gamma_0$, let $\langle \gamma \rangle$ denote the Γ-conjugacy class of γ in Γ. We then claim that if $\theta \in \Psi_r \backslash \{1\}$, then $\theta \in \langle \delta \rangle$ for some $\delta \in \Delta$. We have in fact $b_r \theta b_r^{-1} \in \mathrm{G}\ (12N)$. While $b_{r+1}\Gamma_0 b_{r+1}^{-1} \cap \mathrm{G}\ (12N) = \{1\}$; the second fact shows that $b_{r+1} = x\gamma$ with $x \in B_0$ and $\gamma \in \Gamma$. Further $b_r = h_r^{-1} b_{r+1}$ so that $\delta = \gamma\theta\gamma^{-1} \in \Delta$. Now if $\delta \in \Delta_0$, i.e. if δ is *a good* unipotent it follows that $b_i \theta b_i^{-1} \in U \backslash \{1\} \subset E^*$ and we conclude that

$$p^{-12} \geqslant |b_r \theta b_r^{-1}| \geqslant p^{3N(r-1)} |b_1 \theta b_1^{-1}| \geqslant p^{3N(r-2)} |g\theta g^{-1}|$$

leading to

$$|g\theta g^{-1}| \leqslant p^{-3N(r+2)}.$$

Suppose then that for every $\theta \in \Psi_r \backslash \{1\}$, $\theta \in \langle \delta \rangle$ for $\delta \in \Delta'_0$. In this case let s be the minimal positive integer such that $\Psi_{s+1} \cap \langle \delta \rangle$ has cardinality at most one for any $\delta \in \Delta'_0$. In view of condition 3^0 (in our choice of the h_i) in 3.13, we see that there is an integer $a \geqslant 0$ with $a \leqslant l$ such that $b_{s+1+a}(\Psi_{s+1} \cap \langle \delta \rangle) b_{s+1+a}^{-1} \subset E^*$ for all $\delta \in \Delta'_0$. Let $r - s - 1 - a = t$; then one has for $\theta \in \Psi_r \cap \langle \delta \rangle$ ($\delta \in \Delta'_0$ necessarily) we have

$$p^{-12} \geqslant |b_r \theta b_r^{-1}| \geqslant p^{3Nt} |b_{s+1+a} \theta b_{s+1+a}^{-1}| \geqslant p^{3Nt - N'}$$

by the definition of N'. Thus $N' \geqslant 3Nt + 12$ so that $t < f$. It follows now that $r - s \leqslant e + f$. On the other hand Ψ_s contains two distinct element θ, θ' both belonging to $\langle \delta \rangle$ for some $\delta \in \Delta_0'$. It follows that $\phi = \theta'\theta^{-1}$ is a commutator (in Γ, hence) in G. Evidently it belongs to Ψ_s. Thus Ψ_s contains a good unipotent ϕ. Let s' be the integer ($\geqslant s$) such that $\phi \in \Psi_{s'}$, $\phi \notin \Psi_{s'+1}$. Now for all $x \in \Psi_{s'+1}$, $x \neq 1$, we have

$$\begin{aligned} |b_{s'+1} x b_{s'+1}^{-1}| &= |h_{s'+1}^{-1} h_{s'+2}^{-1} \cdots h_r^{-1} b_{r+1} x b_{r+1}^{-1} h_r h_{r-1} \cdots h_{s'+1}| \\ &\geqslant p^{-6N(r-s')-12N} \geqslant p^{-N_1}. \end{aligned}$$

Thus $b_{s'+1} = x\gamma$ with $x \in B_1$ and $\gamma \in \Gamma_0$. As $b_s, x b_{s'}^{-1} \in \mathrm{G}\,(12N)$ while $h_{s'}^{-1} \in C$, we conclude that $\gamma\phi\gamma^{-1} \in \Delta_1$. Since $b_1 \phi b_1^{-1}$ is in $\mathrm{G}\,(12N)$ and Γ_1 is in good position, $b_1 \phi b_1^{-1} \in U \backslash \{1\} \subset E^*$. Thus we find that

$$p^{-12N} \geqslant |b_{s-1} \phi b_{s-1}^{-1}| \geqslant p^{3N(s-2)} | \geqslant b_1 \phi b_1^{-1}| \geqslant p^{3N(s-3)} |g\phi g^{-1}|$$

leading to $|g\phi g^{-1}| \leqslant p^{-3N(s+1)}$ and since $s \geqslant r - e - f$,

$$|g\phi g^{-1}| < p^{-3N(r-e-f+1)} .$$

Finally let $y \in \Gamma - \{1\}$ be such that $p^{-r'} = |gyg^{-1}| \leqslant |gy'g^{-1}|$ for all $y' \in \Gamma - \{1\}$. Then $|b_1 y b^{-1}| \leqslant p^{-r'N}$ and hence $p^{-12N} \leqslant |b_{r+1} y b_{r+1}^{-1}| \leqslant p^{6rN - r' - N}$ leading to the inequality

$$r' \leqslant 6N(r+2) + N.$$

We have thus proved the following.

Theorem 3.15. *Let $\Gamma \subset G$ be an irreducible L-subgroup. Then there exist integers $l, N_0 > 0$ and a finite set $\Delta_1 \subset \Gamma$ of nontrivial good unipotents such that the following holds. If $g \in G$ is such that $g\Gamma g^{-1} \cap \mathrm{G}(2n + l) \neq \{1\}$ with $n \geqslant N_0$ then there exists $\delta \in \Delta_1$ and $\theta \in \langle \delta \rangle$ such that $|g\theta g^{-1}| < p^{-n}$.*

COROLLARY 3.16

If $P' \subset G$ is a parabolic subgroup of G such that $U' \cap \Gamma \neq \{1\}$, U' being the nilradical of P', then there is a $\delta \in \Delta_1$ and $\gamma \in \Gamma$ such that $P' = \gamma P(\delta) \gamma^{-1}$.

4. Fundamental domains

We will prove the existence of a good fundamental domain for an irreducible L-subgroup $\Gamma \subset G$ in this section. The first step towards this is

Theorem 4.1. *Let Γ be an irreducible L-subgroup of G and $\theta \in \Gamma \backslash \{1\}$ a good unipotent. Let $P(\theta)$ (resp. $U(\theta)$) be the unique minimal parabolic subgroup (resp. horocyclic subgroup) containing θ and ${}^0P(\theta) = \{g \in P \mid g$ normalizes U and preserves a Haar measure on it$\}$. Then ${}^0P(\theta)/{}^0P(\theta) \cap \Gamma$ is compact.*

4.2 After conjugating Γ by a suitable element of G, we may assume that $\theta \in U$. Clearly then $U(\theta) = U$ and $P(\theta) = P$. We also set ${}^0P = {}^0P(\theta)$. Let $q{:}P \to P/U$ be the natural map. Then q maps Z isomorphically onto P/U. Let D be the *unique* maximal pro-p

subgroup of Z. (Such a subgroup exists: this follows from the fact that $Z=\prod_{i\in I} Z_i$ with $Z_i=\mathbf{Z}_i(k_i)$ where $\mathbf{Z}_i$ is a reductive k_i-algebraic group whose commutator subgroup is anisotropic over k_i). Now U admits a family U_n, $n\in\mathbb{Z}^+$, of open compact D-stable subgroups such that $\bigcup_{n\in\mathbb{Z}^+} U_n=U$. The U_n, $n\in\mathbb{Z}^+$ are all pro-p groups and hence so are the DU_n. It follows that $DU_n\cap\Gamma$ is a finite unipotent group for every n and hence $(DU\cap\Gamma)$ is a unipotent group which we denote Λ in the sequel. Let L denote the "Zariski closure" of Λ in G. Here and in the sequel we mean by the *Zariski closure* of a subgroup Γ' of G, the subgroup G' of G is obtained as follows: let $\Gamma_i'(\subset G_i)$ be the projection of Γ' in G_i and $\mathbf{G}_i'$ be the Zariski closure of Γ_i' in $\mathbf{G}_i$; let $G_i'=G_i'(k_i)$, then $G'=\prod_{i\in I} G_i'$ (*Note that all Zariski closures in our definition decompose as products of k_i-points of algebraic subgroups in G_i*). We observe that if $A\subset G$ is any set, the centraliser $Z(A)$ of A is its own Zariski closure; also A admits a *finite* subset A' such that $Z(A)=Z(A')$. Yet another observation needed repeatedly in the sequel is the following: *if $B\subset G$ is any subgroup, then $Z(B)=Z(\bar{B})$ where $\bar{B}$ is the Zariski closure of* B. We record for future use the following well known result.

Lemma 4.3. *If $A\subset\Gamma$ is any subset, the natural map $Z(A)/Z(A)\cap\Gamma\to G/\Gamma$ is proper.*

Proof. Using a standard argument involving the Baire-category Theorem, it is easy to see that we need only prove that $Z(A)\cdot\Gamma$ is closed in G. We assume that A is finite. Suppose $g_n\in Z(A)$ and $\gamma_n\in\Gamma$ are sequences such that $g_n\gamma_n$ converges to a limit. Then for $x\in A$, $\gamma_n^{-1}g_n^{-1}xg_n\gamma_n=\gamma_n^{-1}x\gamma_n$ converges to a limit; but $\gamma_n^{-1}x\gamma_n\in\Gamma$. Thus we see that for some integer $m>0$ we have $\gamma_n^{-1}x_n\gamma_n=\gamma_m^{-1}x\gamma_m$ for all $n\geqslant m$. i.e. $\theta_n=\gamma_n\gamma_m^{-1}\in Z(A)\,\Gamma$ for $n\geqslant m$. Clearly $g_n\theta_n$ converges to a limit. This proves the lemma.

4.4 Now let Φ be a *maximal abelian* subgroup of $U\cap\Gamma$ and set $\Phi'=Z(\Phi)\cap\Gamma$. Evidently $\Phi\subset\Phi'$. Let F' be the Zariski closure of Φ'. We claim that *F'/Φ' is compact.* Since F' is contained in $Z(\Phi)$, it suffices to show that $Z(\Phi)/Z(\Phi)\cap\Gamma$ *is compact.* If this last space is not compact, in view of Lemma 4.3 we can find sequences $g_n\in Z(\Phi)$ and $\gamma_n\in\Gamma\backslash\{1\}$, γ_n a good unipotent such that $g_n\gamma_ng_n^{-1}\to 1$. We assume (as we may after a conjugation by a suitable element of T) that $Z(\Phi\cap\mathrm{G}\,(1))=Z(\Phi)$. Since $g_nxg_n^{-1}=x\in\mathrm{G}\,(1)$ for all $x\in\Phi\cap\mathrm{G}\,(1)$, we conclude that $\mathrm{G}\,(1)\cap\Phi$ and γ_n generate a unipotent subgroup of Γ. By forming repeated commutations of γ_n with elements of $\Phi\cap\mathrm{G}\,(1)$ we see that we can find $\theta_n\in\Phi$ with $\theta_n=g_n\theta_ng_n^{-1}$ converging to identity, a contradiction. Thus $Z(\Phi)/Z(\Phi)\cap\Gamma$ is compact. One consequence of this is that $Z(\Phi)\cap U\subset F'$. This follows from the following two observations: There is a representation ρ of G on a vector space W and a vector $w_0\in W$ such that $F'=\{g\in G\,|\,\rho(g)w_0=w_0\}$; secondly, the orbit of any $v\in W$ under $Z(\Phi)\cap U$ if relatively compact is trivial (these observations are consequences of standard results about *split* unipotent groups over local fields). Since Φ is *abelian* $Z(\Phi)\cap U$ contains a *maximal abelian* subgroup U_1 of U. From the structure of rank 1 algebraic groups over local fields (using classification, for instance) it is not difficult to see that U_1 is *maximal abelian in* G. Thus one has $Z(F')(=Z(\Phi'))\subset U_1$. Since $\Phi'\subset\Lambda$ one sees without difficulty that $Z(\Phi)\subset MU\subset\mathrm{D}$. $U-Z(\Lambda)$ is contained in U_1. Also since Λ is unipotent $Z(\Lambda)$ is non-trivial, the map $Z(\Lambda)/Z(\Lambda)\cap\Gamma\to G/\Gamma$ is proper and factors through the *compact* space $Z(\Phi)/Z(\Phi)\cap\Gamma$, $Z(\Lambda)/Z(\Lambda)\cap\Gamma$ *is compact.* Let $\Phi_0=Z(\Lambda)$ and F_0 the Zariski closure of Φ_0. Then $F_0=Z(\Lambda)$ and F_0/Φ_0 is compact. Also $F_0\subset U$.

4.5 Consider now the case when $|I| = 1$ i.e. $G = G(k')k'$ *a finite extension of k and G a k-rank* 1 *absolutely simple k'-algebraic group of adjoint type.* In this case Λ has *finite* index $P \cap \Gamma$ as is easily seen. Also ${}^0P = MU$ so that $(P \cap \Gamma)\cdot{}^0P = K \cdot U$ where K is a *compact* subset of P. We claim now that $Z(\Phi_0)/Z(\Phi_0) \cap \Gamma$ is compact. If not we can find $g_n \in G$, $\gamma_n \in \Gamma$ and a nontrivial good unipotent $\delta \in \Gamma$ such that $g_n \gamma_n \delta \gamma_n^{-1} g_n^{-1}$ tends to 1. We may assume that $Z(\Phi_0) = Z(\Phi_0 \cap \mathrm{G}(1))$ so that for $x \in Z(\Phi_0) \cap \mathrm{G}(1)$, $x = g_n x g_n^{-1}$ and $g_n \gamma_n \delta_n \gamma_n^{-1} g_n^{-1}$ are both contained in the same unipotent group. It follows that $\gamma_n \delta \gamma_n^{-1} \in U \cap \Gamma$. Thus replacing δ by $\gamma_1 \delta \gamma_1^{-1}$ and γ_n by $\gamma_n \gamma_1^{-1}$, we find that $g_n \gamma_n \delta \gamma_n^{-1} g_n^{-1}$ tends to 1 with $\gamma_n \in P \cap \Gamma$ and $\delta \in U \cap \Gamma$. Now $g_n \gamma_n = k_n \cdot u_n$, $k_n \in K$, $u_n \in V$; K being compact this means that $u_n \delta u_n^{-1}$ tends to 1. But the inner conjugation orbit of δ under the ("split") unipotent group U is closed, a contradiction. This proves our claim that $Z(\Phi_0)/Z(\Phi_0) \cap \Gamma$ is compact. Since $Z(\Phi_0) \supset L \supset Z(\Phi_0) \cap \Gamma = \Lambda$, we see that L/Λ *is compact.* We assert now that $L \supset U$. To see this let N be the normaliser of L in G. Then N normalises $Z(L) = Z(\Lambda) = F_0$. Let $N_0 = \{g \mid g \in N, g$ preserves a Haar measure μ on $F_0\}$. If $L \cap U = U' \neq U$, let $U_1 = \{g \in U \mid gxg^{-1}x^{-1} \in U'$ for all $x \in L\}$; then U_1/U' is a non-compact group. It is clear that U_1 normalises L and – as is easily seen – that $U_1 \subset N_0$. Now the map $N_0/N_0 \cap \Gamma \to G/\Gamma$ *is proper*: this is seen as folows: let $g_n \in N_0$ and $\gamma_n \in \Gamma$ be sequences such that $g_n \gamma_n$ converges to a limit x. We assume (as we may after a conjugation) that $\mu(F_0 \cap \mathrm{G}(1)) > \mu(F_0/\Phi_0)$. Then one can find $\theta_n \in \Phi_0 \backslash \{1\}$ such that $g_n \theta_n g_n^{-1} \in \mathrm{G}(1)$. It follows immediately that $\gamma_n^{-1} \theta_n \gamma_n \in \Gamma \cap A$ where A is a compact set. Thus passing to a subsequence we may assume that $\gamma_n^{-1} \theta_n \gamma_n = \gamma_m^{-1} \theta_m \gamma_m$ for all $n \geqslant m$ for some integer $m > 0$. *But since* $\theta_n \in U$ for all $n \geqslant m$, $\gamma_n \in P \cap \Gamma$. Since $(P \cap \Gamma)/\Lambda$ is finite we see that we can find $\gamma'_n \in \Lambda$ such that $g_n \gamma'_n$ converges to a limit. This shows that $N_0/N_0 \cap \Gamma \to G/\Gamma$ is proper. Clearly $U_1 L/L \approx U_1/U'$ is non-compact so that we can find $u_n \in U_1$, $\gamma_n \in \Gamma$ and a nontrivial good unipotent $\delta \in \Gamma$ such that $u_n \gamma_n \delta \gamma_n^{-1} u_n$ tends to 1. Once again since $u_n \in N_0$, we can find $\theta_n \in \Phi_0$ such that $u_n \theta_n u_n^{-1} \in \mathrm{G}(1)$. We conclude that $\gamma_n \delta \gamma_n^{-1}$ and θ_n generate a unipotent group. Replacing δ by $\gamma_1 \delta \gamma_1^{-1}$ and γ_n by $\gamma_n \gamma_1^{-1}$, we see that $\gamma_n \in P \cap \Gamma$. But then $u_n \gamma_n = k_n \xi_n$ with $k_n \in K$ and $\xi_n \in U$; as before this leads to a contradiction since the U-orbit of δ is closed. This proves that $U' = U$. Thus sine L/Λ is compact ${}^0P/{}^0P \cap \Gamma$ is compact when $G = G(k')$ G an absolutely simple k'-rank 1 k'-algebraic group of adjoint type.

4.6 Suppose now $G = \prod_{i \in I} G_i$. Let G_i be a compact open subgroup of G_i. Let $\eta_i : G \to G_i$ be the cartesian projection and $H_i = \bigcap_{j \neq i} \eta_j^{-1}(\mathrm{G}_j)$. Then H_i is an open subgroup of G. Let $\Gamma'_i = H_i \cap \Gamma$ and Γ_i the projection of Γ'_i on G_i. Then Γ_i is evidently a discrete subgroup of G_i. Since $H_i/\Gamma'_i \to G/\Gamma$ and $H_i \to G_i$ are proper, one sees easily that the Γ_i are L-subgroups of the G_i. It is now clear that if $\Gamma \cap U \neq \{1\}$, then $\Gamma_i \cap U_i \neq \{1\}$ for all i. By the results of 4.6 we know that $M_i U_i / M_i U_i \cap \Gamma_i$ is compact ($M_i = \eta_i(M)$). It is immediate from this that $MU/MU \cap \Gamma$ *is compact.* Now $\Gamma \not\subset P$. In fact if $\Gamma \subset P$, $\Gamma \cap P$ normalises $M \cdot U \cap \Gamma$; since $MU/MU \cap \Gamma$ is compact, $\Gamma \cap P$ preserves a Haar measure on MU so that $\Gamma \subset {}^0P$. Now let $t_n \in T$ be a sequence such that $|t_n x t_n^{-1}| \geqslant p^n |x|$ for $x \in U$. Then $\bar{t}_n =$ image t_n in G/Γ has no convergent subsequence. But then we can find nontrivial good unipotents θ_n such that $t_n \theta_n t_n^{-1}$ tends to 1, a contradiction since $\theta_n \in \Gamma \cap {}^0P$ and hence $\theta_n \in U$. Now let $\theta \in \Gamma \backslash P$. Let $P' = \theta P \theta^{-1}$ and more generally for any subset $A \subset G$, $A' = \theta A \theta^{-1}$. Let $Z^* = P \cap P'$ and T^* the unique conjugate of T in Z_1. From the fact that $MU/MU \cap \Gamma$ is compact, one sees easily that F_0 is contained in the centre of U and is hence T^*-stable. The same applies to F'_0. It is also not difficult to see that if

$t \in T^*$ preserves the Haar measure on F_0 it preserves the Haar measure on F'_0 as well. Let $T_0^* = \{t \in T^* | t$ preserves the Haar measure on $F_0\}$. Now if $t_n \in T_0^*$ is such that image $t_n (= \bar{t}_n)$ in G/Γ has no convergent subsequence we can find $1 \neq \theta_n \in \Gamma \cap U$ such that $t_n \theta_n t_n^{-1}$ tends to 1. On the other hand we can find $\theta'_n \neq 1$ in $F_0 \cap \Gamma$ such that $t_n \theta'_n t_n^{-1}$ is in a fixed compact set. We conclude that θ_n and θ'_n generate a unipotent group, a contradiction since $U' \neq U$. We see thus that $T_0^* \cdot M \cdot U/P \cap \Gamma$ is compact. Thus $T_0^* MU$ is unimodular and one concludes from this that $T_0^* MU = {}^0P$. Thus ${}^0P/{}^0P \cap \Gamma$ is compact. This proves Theorem 4.1.

Theorem 4.7. *Let $\Gamma \subset G$ be an irreducible L-subgroup. Then there is a finite set $\sum$, a constant $t_c \geqslant 0$, a maximal compact subgroup* G^* *and a compact subset $\eta \subset {}^0P$ such that*

$$G = \mathrm{G}^* \cdot A_t \cdot \eta \cdot \textstyle\sum \Gamma, \quad \text{for all } t \geqslant t_c$$

where for $t > 0$ $A_t = \{ |x \in T| |\chi(x)| \leqslant t\}$, χ is the character on T given by $\mathrm{Int}(t)(\mu) = \chi(t)\mu$, *$\mu$ a Haar measure on U. Moreover we have for every $\xi \in \sum$, $\xi^{-1}\eta\xi \cdot (\Gamma \cap \xi^{-1}{}^0P\xi) = \xi^{-1}{}^0P\xi$ and there exists $t_0 > 0$ such that if $\mathrm{G}^* \cdot A_t \eta \xi \gamma \cap \mathrm{G}^* A_{t_0} \eta \xi' \neq \phi$ then $\xi = \xi'$ and $\gamma \in \xi^{-1}{}^0P\ \xi \cap \Gamma$.*

COROLLARY 4.8

If $|I| \geqslant 2$, Γ is finitely generated. This follows from the fact that for any good unipotent $1 = \theta \in \Gamma$, ${}^0P(\theta)/{}^0P(\theta)\ \Gamma$ is compact and ${}^0P(\theta)$ is compactly generated combined with the theorem above. The theorem is proved by a straightforward imitation of the proof of Theorem 13.12 *of* [6].

4.9 From the structure of local fields, we know that we can find a finite field f (of maximal cardinality such that) each of k_i is a finite unramified extension of $f((X))$, the quotient field of the power series ring $f[[X]]$ in one variable over f. (in other words, replacing k by a suitable field we may assume that all the k_i are unramified extensions of k). With this modification we see that $\mathbf{G} = \prod_{1 \leqslant i \leqslant q} \mathbf{G}_i$ may be regarded as a *semisimple* k-algebraic group of adjoint type and that $G = \mathbf{G}(k)$. With these remarks we have thus following corc'lary.

COROLLARY 4.10

If $\Gamma \subset G$ is irreducible, Γ is Zariski dense in the algebraic group $\mathbf{G}$.

Proof. Let $\mathbf{P} \subset \mathbf{G}$ be any minimal k-parabolic subgroup then $\Gamma \not\subset P = \mathbf{P}(k)$). If G/Γ is compact this is clear from the fact that P is *not* unimodular when G/Γ is not compact, this is shown in 4.6. Thus it suffices to show that for some minimal parabolic subgroup $\mathbf{P} \subset \mathbf{G}$, the unipotent radical $\mathbf{U}$ of $\mathbf{P}$ is in the Zariski closure of Γ in G. Theorem 4.7 assures us of this when G/Γ is not compact. When G/Γ is *compact* we argue as follows. Let $\mathbf{H}$ be the Zariski closure of Γ in $\mathbf{G}$ and $H = \mathbf{H}(k)$. Let $\rho: G \to PGL(E)$ be a k-representation of G in $PGL(E)$, E a k-vector space such that $H = \{g \in G | \rho(g)(p) = p\}$ for a suitable k-point p of the projective space $\mathbb{P}(E)$. The closure of the U-orbit $\Lambda(U = \mathbf{U}(k))$ of p contains a $\mathbf{U}$-fixed point since $\mathbf{U}$ is k-split. But Λ is contained in the G orbit of p and this G-orbit is compact. Thus we find that U has a fixed point in G/H. This means that a conjugate of U is contained in H. This proves the corollary.

COROLLARY 4.11

Let $G=\prod_{i\in I} G_i$ $(I=(1,2,\ldots q))$ be as above and Γ any L-subgroup of G. Then there is a partition $I=I_1 \cup I_2 \cdots \cup I_r$ of I into disjoint subsets I_j, $1\leqslant j\leqslant r$ such that if we set $H_j=\prod_{i\in I_j} G_i$ and $\Gamma_j=H_j=H_j\cap\Gamma$ then Γ_j is an irreducible L-subgroup of H_j and $\prod_{1\leqslant j\leqslant r}\Gamma_j$ has finite index in Γ.

Proof. We will argue by induction on the number of factors of G. Let $\mathsf{I}=\{J\subsetneqq I\,|\,\Gamma\cap\prod_{i\in J} G_i\neq\{1\}\}$. If I is empty Γ is irreducible and there is nothing to prove. Assume that $\mathsf{I}\neq\phi$ and let $\mathsf{I}'\in I$ be a minimal element. Let $G'=\prod_{i\in I'} G_i$ and $G''=\prod_{i\in I\setminus I'} G_i$. Let $\pi\colon G'\times G''(=G)\to G'$ be the cartesian projection. Let $\Gamma'_0=\Gamma\cap G'$ and $\tilde{\Gamma}'=\pi(\Gamma)$. Evidently Γ'_0 is a normal subgroup $\tilde{\Gamma}'$ (note that $\Gamma'_0=\pi(\Gamma'_0)$). Let K'' be a maximal compact open subgroup and $\Gamma'=\pi(\Gamma\cap(G'\times K''))$. Evidently $\Gamma'_0\subset\Gamma'\subset\tilde{\Gamma}'$ and Γ' is easily seen to be a L-subgroup of G'. We claim that Γ' *is an irreducible L-subgroup in* G'. Suppose that Γ' is not irreducible. Since $|I'|<|I|$, by induction hypothesis, G' decomposes as a product $\prod_{\alpha\in A} H_\alpha$ such that $H_\alpha\cap\Gamma'=\Gamma'_\alpha$ is a irreducible L-subgroup of H_α and $\prod_{\alpha\in A}\Gamma'_\alpha=\Gamma'_1$ is of finite index in Γ'. By Corollary 4.10, Γ'_α is Zariski dense in H'_α. Clearly Γ'_α normalises Γ'_0. Hence the Zariski closure H'_0 of Γ'_0 is normalised by H'_α. As this holds for every α, H'_0 is a normal subgroup of G'. Thus H'_0 is a product of certain of the k-simple factors of G' and from the minimality of I', one sees that $H'_0=G$. But then $[\Gamma'_\alpha,\Gamma'_0]$ is nontrivial and contained in H_α, a contradiction to the minimality of I' if $H_\alpha\neq G'$. Thus Γ' is an irreducible L-subgroup in G'. Moreover since Γ'_0 is Zariski dense in G', it is immediate that the normaliser of Γ'_0 in G' is discrete. We conclude from this that $\tilde{\Gamma}'$ is discrete and Γ' has finite index in $\tilde{\Gamma}'$. But this means that $\Gamma G''$ is closed in G or in other words, the map $G''/G\cap\Gamma\to G/\Gamma$ is *proper*. If $g_n\in G''$ is a sequence tending to infinity mod $\Gamma''_0=G''\cap\Gamma$, it follows that we can find $\theta_n\in\Gamma\setminus\{e\}$ such that $g_n\theta_n g_n^{-1}$ tends to identity. On the other hand for $\rho\in\Gamma'_0$, $g_n\rho g_n^{-1}=\rho$; thus $g_n(\rho\theta_n\rho^{-1}\theta_n^{-1})g_n^{-1}$ tends to the identity; since $\rho\theta_n\rho^{-1}\theta_n^{-1}\in G'$ and G' and G'' commute we see that $\rho\theta_n\rho^{-1}\theta_n^{-1}$ tends to 1 i.e. $\rho\theta_n\rho^{-1}\theta_n^{-1}=1$ for large n. Varying ρ over a suitable finite set and using the Zariski density of Γ'_0 in G', we conclude that θ_n commute with G' for large n i.e. $\theta_n\in G''\cap\Gamma=\Gamma''_0$ for large n. Thus Γ''_0 is a L-subgroup of G''. The induction hypothesis combined with Corollary 4.10 now shows that Γ''_0 is Zariski dense in G''. Thus $G'=Z(\Gamma''_0)$, the centraliser of Γ''_0 in G so that $G'/G'\cap\Gamma\to G/\Gamma$ is proper. One now concludes arguing as above (with Γ''_0 in place of Γ'_0) that $\Gamma'_0=G'\cap\Gamma$ is a L-subgroup of G'. It is now clear that $\Gamma'_0\times\Gamma''$ is a L-subgroup of G and has finite index in Γ. Using the induction hypothesis on G' and G'', the result now follows for G.

5. Appendix

We will prove the following using the notations of §2.

PROPOSITION 5.1

The natural map ω: $\mathbf{U}\to\mathbf{U}/\mathbf{U}'$ *admits a section* $\theta\colon\mathbf{U}/\mathbf{U}'\to\mathbf{U}$ *defined over k compatible with the action of* $\mathbf{T}$.

Proof. Since $\mathbf{U}/\mathbf{U}'$ is affine and $\mathbf{U}'$ is a vector space over k, the fibration ω is trivial. Thus we can find a section $\rho\colon\mathbf{U}/\mathbf{U}'\to\mathbf{U}$ *defined over k*. If ρ_1, ρ_2 are two sections to

ω one has a morphism α: $\mathbf{U}/\mathbf{U}' \to \mathbf{U}'$ such that $\rho_1(x) = \rho_2(x)\cdot\alpha(x)$. If ρ_1, ρ_2 are defined over k so is α. Let $t \in T$; then one has $t^{-1}\rho(txt^{-1})t = \rho(x)\cdot\phi(t, x)$ where $\phi(t, -)$ is a morphism of $\mathbf{U}/\mathbf{U}'$ in $\mathbf{U}'$ denoted $\Phi(t)$ in the sequel. It is easy to to see that $\phi: \mathbf{T} \times \mathbf{U}/\mathbf{U}' \to \mathbf{U}'$ is a k-morphism and that Φ: $\mathbf{T} \to \mathrm{Hom}(\mathbf{U}/\mathbf{U}', \mathbf{U}')$ (where Hom denotes the morphisms in the category of algebraic varieties) is a 1-cocycle on $\mathbf{T}$. Here $\mathfrak{H} = \mathrm{Hom}\,(\mathbf{U}/\mathbf{U}', \mathbf{U}')$ is given the abelian group structure derived from that on $\mathbf{U}'$ and $t \in \mathbf{T}$ acts on $\mathfrak{H}$ by $f \to \overline{\mathrm{Int}}\ \mathrm{t.f.Int\,t}^{-1}$ where $\mathrm{Int}\,t$ is the inner automorphism of $\mathbf{G}$ induced by t and $\overline{\mathrm{Int}}$ is the natural automorphism of $\mathbf{U}/\mathbf{U}'$ induced by $\mathrm{Int}\,t$. Let $\mathfrak{H}(n) = \{f \in \mathfrak{H} \mid f$ is homogeneous of degree $n\}$ (Note that $\mathbf{U}'$ and $\mathbf{U}/\mathbf{U}'$ are k-vector spaces in a natural fashion). Then $\mathbf{T}$ acts through the character 2α-$n\alpha$ on $\mathfrak{H}(n)$. It follows that $H^1(T, \mathfrak{H}(n)) = 0$ for $n \neq 2$ (see for instance Raghunathan [6, Preliminaries]). If $n = 2$ $H^1(T, \mathfrak{H}(\mathrm{n}))$ will consist of abstract group homomorphisms of $\mathbf{T}$ in $H(n)$. Now $\Phi: \mathbf{T} \to \mathfrak{H}$ necessarily factors through to an algebraic morphism of $\mathbf{T}$ into a finite dimensional k-vector space $\bigsqcup_{0 \leqslant n \leqslant N} \mathfrak{H}(n)$ for some integer $N \geqslant 2$. Let $\Phi_n = p_n \circ \Phi$ where p_n is the natural projection on $\mathfrak{H}(n)$. Then since Φ_2 is algebraic, it is zero. On the other hand for $n \neq 2$, Φ_n is the coboundary of a *unique* $f_n \in \mathfrak{H}(n)$. Since Φ_n is defined over k, so is f_n as is easily seen. Clearly $\{f_n\}_{0 \leqslant n \leqslant N}$ (with $f_2 = 0$) define a morphism f of $\mathbf{U}/\mathbf{U}'$ in $\mathbf{U}'$. If we now modify ρ by f we obtain a section θ satisfying the requirements of the lemma.

References

[1] Behr H, Eindliche Erzeugbarkeit arithmetischer Gruppen über Funktionenkorper, *Inv. Math.* **7** (1969) 1–32

[2] Borel A and Tits J, Groupes réductifs, *Publ. Math. Inst. Haut. Sci.* **27** (1965) 55–150

[3] Borel A and Tits J, Eléments unipotents et sousgroupes paraboliques de groupes réductifs, *Inv. Math.*12 (1971), 95–104

[4] Harder G, Minkowskische Reduktionstheorie über Funktionenkorper, *Inv. Math.* **7** (1969) 33–54

[5] Kazhdan D A and Margulis G A, Proof of Selberg's hypothesis, *Math. Sbornik* (N.S) **75** (117) (1968), 162–168 (Russian)

[6] Raghunathan M S, Discrete subgroups of Lie Groups, (New York, Heidelberg: Springer Verlag) (1972)

[7] Raghunathan M S, Discrete groups and Q-structures on semisimple groups in Discrete Subgroups of Lie Groups and Applications to Moduli, (Bombay: Oxford University Press) (1975) 225–321

[8] Tits J, Classification of algebraic semisimple groups, *Proc. Symp. Pure Math.* **9** (1969) 33–62

[9] Venkataramana T N, On super-rigidity and arithmeticity of lattices in semisimple groups, *Inv. Math.* **92** (1988) 255–306

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Small fractional parts of additive forms

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Abstract. Let $f(\mathbf{x})=\theta_1 x_1^k+\cdots+\theta_s x_s^k$ be an additive form with real coefficients, and $\|\alpha\|=\min\{|\alpha-u|: u\in\mathbb{Z}\}$ denote the distance from α to the nearest integer. We show that if $\theta_1,\ldots,\theta_s$ are algebraic and $s=4k$ then there are integers $x_1,\ldots,x_s$ satisfying $1\leqslant x_i\leqslant N$ and $\|f(\mathbf{x})\|\leqslant N^E$, with $E=-1+2/e$.

When $s=\lambda k$, $1\leqslant\lambda\leqslant 2k$, the exponent E may be replaced by $\lambda E/4$, and if we drop the condition that $\theta_1,\ldots,\theta_s$ be algebraic then the result holds for almost all values of $\boldsymbol{\theta}\in\mathbb{R}^s$. When $k\geqslant 6$ is small a better exponent is obtained using Heath-Brown's version of Weyl's estimate.

Keywords. Additive forms; Heath-Brown's version; Weyl's estimate; fractional parts.

1. Introduction

Dirichlet's theorem on Diophantine approximation states that for any $\theta\in\mathbb{R}$, $N>1$ there exists $n\in\mathbb{N}$ satisfying $1\leqslant n\leqslant N$ and

$$\|\theta n\|\leqslant N^{-1}.$$

Analogous results for $\|\theta n^k\|$ were obtained by Heilbronn [8] (for the case $k=2$) and Danicic [4]. Their results show that for any $\varepsilon>0$ and $N\geqslant N(\varepsilon,k)$ there exists $n\leqslant N$ satisfying

$$\|\theta n^k\|\leqslant N^{-1/K+\varepsilon}, \tag{1}$$

where $K=2^{k-1}$. The result depends on Weyl's method for estimating exponential sums [13]; as one might expect from the appearance of K in the exponent.

Recently Heath-Brown [6, 7] has obtained an improved version of Weyl's inequality, and used it to show that for any $\theta\in\mathbb{R}$, $\varepsilon>0$ and $k\geqslant 6$ there are infinitely many $n\in\mathbb{N}$ with

$$\|\theta n^k\|\leqslant N^{-4/3K+\varepsilon}. \tag{2}$$

Davenport [5] asked whether it was possible to improve on (1) using Vinogradov's estimates. This was done by the author [2], and subsequently the estimates have been improved considerably (see Baker [1]). Karatsuba [9] provided an alternative to Vinogradov's Mean Value Theorem, and used it to study the function $G(n)$ in Waring's problem. Heath-Brown [7] adapted Karatsuba's techniques to show that for any

$\theta \in \mathbb{R}$, $\varepsilon > 0$ and integer $k \geqslant 2$ there are infinitely many $n \in \mathbb{N}$ with

$$\|\theta n^k\| \leqslant n^{-\eta+\varepsilon}, \tag{3}$$

where

$$\eta = \max_{T \in \mathbb{N}} \frac{k}{4T^2}\left(1 - 2\left(\frac{k-1}{k}\right)^T\right). \tag{4}$$

In particular, it followed that there are infinitely many $n \in \mathbb{N}$ with

$$\|\theta n^k\| \leqslant n^{-1/16k}. \tag{5}$$

Heath-Brown combined this result with Roth's Theorem [11] to show that if θ is algebraic then there exists $N(\theta, k)$ such that for any $N \geqslant N(\theta, k)$ there is an $n \leqslant N$ satisfying

$$\|\theta n^k\| \leqslant N^{-1/16k}. \tag{6}$$

In [3] the author considered an additive form of degree k in s variables,

$$f(\mathbf{x}) = f(x_1, \ldots, x_s) = \theta_1 x_1^k + \cdots + \theta_s x_s^k \tag{7}$$

with real coefficients $\theta_1, \ldots, \theta_s$ and showed that for $\varepsilon > 0$, $N \geqslant N(\varepsilon, k)$ and $1 \leqslant s \leqslant K$ there are integers $x_1, \ldots, x_s$, not all zero, satisfying $\max|x_i| \leqslant N$ and

$$\|f(\mathbf{x})\| \leqslant N^{-s/K+\varepsilon}. \tag{8}$$

We note that when $s = 2^{k-3}$ the exponent in (8) is $-1/4+\varepsilon$. Schmidt [12, Theorem 20B] showed that in the case $k=2$, if the coefficients θ_i are not well approximable, then the exponent can be replaced by a function $c(s) \sim -\sqrt{2s}$ as $s \to \infty$. The condition "not well approximable" is satisfied by almost all $(\theta_1, \ldots, \theta_s) \in \mathbb{R}^s$, and in particular if $\theta_1, \ldots, \theta_s$ are algebraic numbers such that $1, \theta_1, \ldots, \theta_s$ are linearly independent over $\mathbb{Q}$.

The purpose of this note is to combine Heath-Brown's result with the techniques used in the case of additive forms. We begin with a Weyl-type of result.

Theorem 1. *Let $k \geqslant 6$ be an integer, $s \leqslant 2^{k-3}$ and suppose first that $\theta_1, \ldots, \theta_s$ are algebraic numbers. For any $\varepsilon > 0$ there exists $N(\varepsilon, k, \boldsymbol{\theta})$ such that for any $N \geqslant N(\varepsilon, k, \boldsymbol{\theta})$ there are integers $x_1, \ldots, x_s$ satisfying $1 \leqslant x_i \leqslant N$ and*

$$\|f(\mathbf{x})\| \leqslant N^{-4s/3K+\varepsilon}. \tag{9}$$

Further, if we drop the restriction that $\theta_1, \ldots, \theta_s$ be algebraic, then the result is true for almost all $\boldsymbol{\theta} \in \mathbb{R}^s$.

Our second result uses Karatsuba-type results to show that with $s = O(k)$ variables we can obtain a negative exponent which is an absolute constant. The choice $s = 4k$ is made for simplicity and the exponent obtained can be improved with other choices of s, we give further details at the end of the paper.

Theorem 2. *Let $s = 4k$ and suppose first that $\theta_1, \ldots, \theta_s$ are algebraic numbers. There exists $N(k, \boldsymbol{\theta})$ such that for any $N \geqslant N(k, \boldsymbol{\theta})$ there are integers $x_1, \ldots, x_s$ satisfying*

$1 \leqslant x_i \leqslant N$ and

$$\|f(\mathbf{x})\| \leqslant N^E \tag{10}$$

with $E = -1 + 2/e = -0{\cdot}264\cdots$ Further, if we drop the condition that $\theta_1, \ldots, \theta_s$ be algebraic then the result holds for almost all $\boldsymbol{\theta} \in \mathbb{R}^s$.

Theorem 2 holds for $k \leqslant 6$ by choosing $s = 4k$ in our earlier estimate (8). For $k \geqslant 7$ the exponent that arises in the proof is

$$E_k = -1 + 2(1 - k^{-1})^k \to -1 + 2e^{-1} \text{ as } k \to \infty. \tag{11}$$

We see that $E_7 = -0{\cdot}320\ldots$, $E_8 = -0{\cdot}312\ldots$ and $E_k \uparrow E$ as k increases. We assume that the reader is familiar with the arguments of Heath-Brown [7], but it will be necessary to repeat some of the details of his work.

2. A preliminary lemma

The following Lemma, which is used to show that a sequence y_n does not take values near to integers, is based on a remark of H L Montgomery. The case when $a_n = 1$ for all n is Theorem 2.2 of Baker [1], Heath-Brown [7] remarked that the more general case follows on taking repetitions of some terms y_n.

Lemma 1. *Let y_n be a real sequence such that*

$$\|y_j\| \geqslant H^{-1} \text{ for } j = 1, \ldots, N. \tag{12}$$

Then for any non-negative integer weights a_n we have

$$\sum_{h=1}^{H} \left| \sum_{n=1}^{N} a_n e(hy_n) \right| > \frac{1}{6} \sum_{n=1}^{N} a_n. \tag{13}$$

We take $\varepsilon > 0$ to be a fixed given number, $\delta > 0$ is an arbitrary number required to make various inequalities correct. The value of δ may vary by a numerical factor on its different appearances. We use Vinogradov's $\ll$-notation where the implicit constants may depend on ε, k and δ.

We observe that if any $\theta_i = a/q \in \mathbb{Q}$ then both theorems follow readily on taking x_i to be a multiple of q. We may therefore suppose that all the coefficients θ_i are irrational. We shall choose a value of i for which a corresponding exponential sum is large and consider the convergents a_m/q_m in the continued fraction for the (irrational) θ_i. The convergents satisfy (see [10])

$$\left| \theta_i - \frac{a_m}{q_m} \right| \leqslant \frac{1}{q_m q_{m+1}}. \tag{14}$$

If θ_i is algebraic then, by Roth's Theorem [11], for any $\delta > 0$

$$\left| \theta_i - \frac{a_m}{q_m} \right| \geqslant \frac{1}{q_m^{2+\delta}} \tag{15}$$

and so

$$q_{m+1} \leqslant q_m^{1+\delta}. \tag{16}$$

Indeed, for almost all θ_i, (15) (and hence (16)) holds for all but a finite number of m (see Theorem 32 of Khinchin [10]). Thus provided the denominator $q_m \geqslant q^*(\theta_i)$ we may suppose that (16) holds for almost all θ_i.

3. Proof of theorem 1

We take $y_n = f(\mathbf{x})$ in Lemma 1. As $x_1, \dots, x_s$ run through $1, 2, \dots, N$ so n runs through $1, \dots, N^s$. Suppose that for $n = 1, \dots, N^s$ we have, for some $H \leqslant N$,

$$\| f(\mathbf{x}) \| = \| y_n \| \geqslant H^{-1}. \tag{17}$$

Taking all the weights a_n to be 1, Lemma 1 shows that

$$\begin{aligned} \tfrac{1}{6} N^s &< \sum_{h \leqslant H} \left| \sum_{x_1=1}^{N} \cdots \sum_{x_s=1}^{N} e(h(\theta_1 x_1^k + \cdots + \theta_s x_s^k)) \right| \\ &= \sum_{h \leqslant H} \left| \prod_{i=1}^{s} \sum_{x=1}^{N} e(\theta_i h x^k) \right|. \end{aligned} \tag{18}$$

Thus for some i with $1 \leqslant i \leqslant s$ we have, using Hölder's inequality,

$$\begin{aligned} \tfrac{1}{6} N^s &< \sum_{h \leqslant H} \left| \sum_{x=1}^{N} e(\theta_i h x^k) \right|^s \\ &\leqslant \left\{ \sum_{h \leqslant H} \left| \sum_{x=1}^{N} e(\theta_i h x^k) \right|^{K/4} \right\}^{4s/K} H^{1-4s/K} \end{aligned} \tag{19}$$

since $s \leqslant 2^{k-3} = K/4$, where $K = 2^{k-1}$.

Heath-Brown [6, Lemmas 1 and 5] has shown that for any $\delta > 0$

$$\left| \sum_{x=1}^{N} e(\beta x^k) \right|^{K/4} \ll N^{K/4-1} + N^{(K/4)-k+2+\delta} \sum_{u=1}^{\kappa N^{k-3}} T(\beta u) \tag{20}$$

where $\kappa = k!\, K/24$ and

$$T(x) = \max \left| \sum_{n \in I} e(yn^3 + zn) \right|, \tag{21}$$

the maximum being taken over subintervals I of $[1, N]$, real numbers

$$y \in [N^{-3}[N^3 x], \quad N^{-3}([N^3 x] + 1)] \tag{22}$$

and $z \in [0, 1]$. The sum T satisfies, for any $\delta > 0$,

$$\sum_{m=1}^{N} T(mN^{-3})^6 \ll N^{7+\delta}. \tag{23}$$

From (19) and (20) we obtain

$$N^{K/4}H^{1-K/4s} \ll HN^{K/4-1} + N^{(K/4)-k+2+\delta} \sum_{h \leqslant H} \sum_{u=1}^{\kappa N^{k-3}} T(\theta_i hu). \tag{24}$$

The first term on the right of (24) is smaller than the terms on the left so, using Hölder's inequality,

$$\begin{aligned} N^{k-2-\delta}H^{1-K/4s} &\ll \sum_{h \leqslant H} \sum_{u=1}^{\kappa N^{k-3}} T(\theta_i hu) \\ &\ll \left\{ \sum_{h \leqslant H} \sum_{u=1}^{\kappa N^{k-3}} T^6(\theta_i hu) \right\}^{1/6} (HN^{k-3})^{5/6} \end{aligned} \tag{25}$$

and hence

$$N^{k+3-\delta}H^{1-3K/2s} \ll \sum_{h \leqslant H} \sum_{u=1}^{\kappa N^{k-3}} T^6(\theta_i hu). \tag{26}$$

From (23) we obtain

$$\sum_{h \leqslant H} \sum_{u=1}^{\kappa N^{k-3}} T^6(\theta_i hu) \ll N^{7+\delta} R \tag{27}$$

where

$$R = \max_{m \leqslant N^3} |\{v \leqslant \kappa HN^{k-3} : \| \theta_i v - mN^{-3} \| \leqslant N^{-3}\}|. \tag{28}$$

For any given $N \geqslant N(\varepsilon, k, \boldsymbol{\theta})$ we can choose a convergent a/q to θ_i satisfying

$$q^{1/3} \leqslant N \leqslant q^{(1+\delta)/3} \tag{29}$$

and take $N_1 = [q^{1/3}]$, obtaining (27) and (28) with N replaced by N_1. Using Lemma 6 of Heath-Brown [6] with the value $N_1 = [q^{1/3}]$ the value of R in (28) is estimated as

$$R \ll HN_1^{k-6} \tag{30}$$

for $k \geqslant 6$. Substituting into (26) we obtain (since $N \leqslant N_1^{1+\delta}$)

$$H^{3K/2s} \gg N^{2-\delta} \tag{31}$$

which completes the proof of Theorem 1.

4. Proof of theorem 2

We use the same weights a_n (or $a(n)$) as Heath-Brown [7]. The essential properties are that, for any fixed positive integers T and J

$$S = \sum_{n=1}^{N} a_n e(\beta n^k) \tag{32}$$

satisfies ([6, eq. 5.3])

$$|S|^{2T^2} \leqslant P^{4T^2-2k+2k\theta T}\left\{\sum_{|d|\leqslant TP^k} \min(TP^k, \|\beta d\|^{-1})\right\} \tag{33}$$

where

$$\theta = 1 - 1/k \quad \text{and} \quad P = N^{1/2}. \tag{34}$$

The reader should note the difference between θ without a suffix, i.e. the parameter $1-1/k$, and θ_i with a suffix, a coefficient of f. Also ([6, eq. 5.5])

$$\sum_{n=1}^{N} a_n \gg N^{1-\theta^J-\delta}. \tag{35}$$

We will apply these estimates with $T=k$ and will take $s=4k$.

Suppose that for $1 \leqslant x_i \leqslant N$, $1 \leqslant i \leqslant 4k$, we have

$$\|f(\mathbf{x})\| \geqslant H^{-1}, \tag{36}$$

where $H \leqslant N$. Defining the weight

$$A(\mathbf{x}) = A(x_1,\ldots,x_s) = a(x_1)a(x_2)\ldots a(x_s). \tag{37}$$

Lemma 1 gives

$$\begin{aligned} N^{4k(1-\theta^J-\delta)} &\ll \sum_{\mathbf{x}} A(\mathbf{x}) = \left(\sum_{n=1}^{N} a_n\right)^{4k} \\ &\leqslant \sum_{h\leqslant H}\left|\sum_{x_1=1}^{N}\cdots\sum_{x_s=1}^{N} a(x_1)\cdots a(x_s)e(h(\theta_1 x_1^k+\cdots+\theta_s x_s^k))\right| \\ &= \sum_{h\leqslant H}\prod_{i=1}^{4k}\left|\sum_{x=1}^{N} a(x)e(h\theta_i x^k)\right|. \end{aligned} \tag{38}$$

Then for some i

$$\begin{aligned} N^{4k(1-\theta^J-\delta)} &\ll \sum_{h\leqslant H}\left|\sum_{x=1}^{N} a(x)e(h\theta_i x^k)\right|^{4k} \\ &\leqslant \left\{\sum_{h\leqslant H}\left|\sum_{x=1}^{N} a(x)e(h\theta_i x^k)\right|^{2k^2}\right\}^{2/k} H^{1-2/k}, \end{aligned} \tag{39}$$

from Hölder's inequality.

Since $T=k$ and $P=N^{1/2}$, (33) gives

$$\begin{aligned} N^{4k^2(1-\theta^J-\delta)}H^{2-k} &\ll \left(\sum_{k\leqslant H}\left|\sum_{x=1}^{N} a(x)e(h\theta_i x^k)\right|^{2k^2}\right)^2 \\ &\ll N^{4k^2-2k+2k\theta^k}\left(\sum_{h\leqslant H}\sum_{|d|\leqslant kP^k}\min(P^k, \|\theta_i hd\|^{-1})\right)^2. \end{aligned} \tag{40}$$

The number of representations of any non-zero $v \leqslant HkP^k$ as $v = hd$ is $O(N^\delta)$ and so

$$N^{2k-2k\theta^k-4k^2\theta^J-\delta}H^{2-k} \ll \left\{HP^k + \sum_{v=1}^{HkP^k}(P^k, \|\theta_i v\|^{-1})\right\}^2. \tag{41}$$

For any given N we can choose a convergent a/q to θ_i with

$$q^{2/k} \leqslant N \leqslant q^{2(1+\delta)/k}, \tag{42}$$

and take $N_2 = [q^{2/k}]$. Then Lemma 6 of Heath-Brown [6] gives

$$\sum_{v=1}^{Hkp^k} \min(P^k, \|\theta_i v\|^{-1}) \ll HP^{k+\delta} \tag{43}$$

since $N \leqslant N_2^{1+\delta}$. Substituting into (41) we have

$$H^k \gg N^{k-2k\theta^k-4k^2\theta^J-\delta} \tag{44}$$

or

$$H \gg N^{1-2\theta^k-4k\theta^J-\delta}. \tag{45}$$

This holds for any positive integer J, and we choose J so large that $4k\theta^J < \delta$, and this completes the proof of Theorem 2 with an exponent $N^{-\Omega}$ for any

$$\Omega < 1 - 2(1-1/k)^k = -E_k. \tag{46}$$

We remark that it is indeed straightforward to adapt the above proof to other values of s. If we take $s = \lambda k$, where $1 \leqslant \lambda \leqslant 2k$, the proof goes through with $H = N^{\Omega}$ for any

$$\Omega < \lambda(1-2\theta^k)/4 = -\lambda E_k/4. \tag{47}$$

Thus with $s = \lambda k$ we can replace (10) by

$$\|f(\mathbf{x})\| \leqslant N^{\lambda E/4}. \tag{48}$$

In particular, taking $s = 2k^2$ we obtain an exponent

$$kE/2 = E\sqrt{s}/(2\sqrt{2}), \tag{49}$$

which may be compared to Schmidt's result for quadratic forms.

References

[1] Baker R C, *Diophantine Inequalities. London Math. Soc. Monographs* N.S.1. (Oxford: Science Publishers) (1986)

[2] Cook R J, On the fractional parts of a set of points, *Mathematika* **19** (1972) 63–68

[3] Cook R J, The fractional parts of an additive form, *Proc. Camb. Philos. Soc.* **72** (1972) 209–212

[4] Danicic I, *Contributions to number theory*, Ph.D. Thesis (Univ. London 1957)

[5] Davenport H, On a theorem of Heilbronn, *Q. J. Math. Oxford* **18** (1967) 339–344

[6] Heath-Brown D R, Weyl's inequality, Hua's inequality, and Waring's problem, *J. London Math. Soc.* **38** (1988) 216–230

[7] Heath-Brown D R, The fractional parts of αn^k. *Mathematika* **35** (1988) 28–37

[8] Heilbronn H, On the distribution of the sequence $n^2\theta$ (mod 1). *Q. J. Math. Oxford* **19** (1948) 249–256

[9] Karatsuba A A, On the function $G(n)$ in Waring's problem. *Izv. Akad. Nauk. SSSR, Ser. Mat.* **49** (1985) 935–947

[10] Ya A Khinchin, *Continued fractions* (Chicago: University Press) (1964)

[11] Roth K F, Rational approximations to algebraic numbers, *Mathematika* **2** (1955) 1–20

[12] Schmidt W M, *Small fractional parts of polynomials*, CBMS Regional Conferences in Mathematics 32 (American Math. Soc 1977)

[13] Weyl H, Über die Gleichverteilung von Zahlen mod Eins. *Math. Ann.* **77** (1916) 313–352

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A generalization of the Riemann zeta-function

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Abstract. A generalization of the Riemann zeta-function which has the form

$$\zeta_a(s) = \prod_p \frac{1}{1 - p^{-s} + (p+a)^{-s}}$$

is considered. Analytical properties with respect to s and asymptotic behaviour when $a \to \infty$ are investigated. The corresponding L-function is also discussed. This consideration has an application in the theory of p-adic strings.

Keywords. Riemann zeta-function.

1. Introduction

The Riemann zeta-function in terms of the Euler product has the form

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \tag{1}$$

where p ranges over all primes ($p = 2, 3, 5, \ldots$) and $s = \sigma + it$, $\sigma > 1$. The importance of $\zeta(s)$ in the number theory is well-known, see for example [1]. In this note we consider a generalization of $\zeta(s)$ defined by

$$\zeta_a(s) = \prod_p \frac{1}{1 - p^{-s} + (p+a)^{-s}} \tag{2}$$

where $a > 0$. Usefulness of such a generalization in the context of the p-adic string theory was noted in Areféva *et al* 1988 (this reference also contains references to the corresponding physical papers).

It will be proved here that the product (2) is absolutely convergent for $\sigma > 0$ and it defines an analytical function which does not vanish for $\sigma \geqslant \sigma_0 > 0$ and large a, depending on σ_0. So the function $\zeta_a(s)$ can be considered as a holomorphic approximation to the Riemann zeta-function. One can hope to get an information on the zeros of the Riemann zeta-function investigating the behaviour of $\zeta_a(s)$ when $a \to \infty$ in the critical strip $0 < \sigma < 1$ (compare density theorem in [1]). It will be shown that the asymptotic behaviour $\zeta_a(1)$ when $a \to \infty$ coincides with the one for $\prod_{p<a} (1 - p^{-1})^{-1}$.

We discuss here also some problems whose solutions would be useful for the p-adic string theory. Finally a generalization of the L-function along the same line is suggested.

2. Properties of $\zeta_a(s)$

The first lemma states that for large a the denominator in eq. (2) has no zeros.

Lemma 1. *Let the following conditions for s and a be satisfied.*

Either 1) $\sigma \geqslant 1, \quad t \in \mathbb{R}, \quad a > 0,$

or 2) $1 > \sigma > \sigma_0, \quad t \in \mathbb{R}$ *and* $a > 2(2^{\sigma_0} - 1)^{-1/\sigma_0}$

for a fixed constant σ_0, $0 < \sigma_0 < 1$.

Then

$$\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right| < 1.$$

Proof. We have

$$\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right| \leqslant \left|\frac{1}{p^s}\right| + \left|\frac{1}{(p+a)^s}\right| = \frac{1}{p^\sigma} + \frac{1}{(p+a)^\sigma} \leqslant \frac{1}{2^\sigma} + \frac{1}{(2+a)^\sigma}.$$

Then for $\sigma \geqslant 1$ one has

$$\frac{1}{2^\sigma} + \frac{1}{(2+a)^\sigma} < 1.$$

If $a > 2(2^{\sigma_0} - 1)^{-1/\sigma_0}$ we have

$$\frac{1}{2^{\sigma_0} - 1} < \left(\frac{a}{2}\right)^{\sigma_0} < \left(\frac{a}{2}\right)^{\sigma_0}\left(1 + \frac{2}{a}\right)^{\sigma_0} = \left(\frac{a}{2} + 1\right)^{\sigma_0} < \left(\frac{a}{2} + 1\right)^\sigma$$

and taking into account

$$\frac{1}{2^\sigma - 1} \leqslant \frac{1}{2^{\sigma_0} - 1}$$

one gets

$$\frac{1}{2^\sigma - 1} < \left(\frac{a}{2} + 1\right)^\sigma \quad \text{or} \quad 1 + \frac{1}{\left(1 + \frac{a}{2}\right)^\sigma} < 2^\sigma$$

or finally

$$\frac{1}{2^\sigma} + \frac{1}{(2+a)^\sigma} < 1.$$

eorem 1. *Let s and a be as in Lemma* 1. *Then the product* (2) *is absolutely convergent.* *is convergence is uniform on compact subsets of the s region and defines an analytic* *ction* $\zeta_a(s)$ *which has no zeros for* $\sigma \geqslant \sigma_0$.

oof. We have

$$\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right| = \left|s\int_0^a \frac{du}{(p+u)^{s+1}}\right| \ll \frac{|as|}{p^{\sigma+1}} \tag{3}$$

en $p \to \infty$.

Let $y_p = y_p(s,a) = p^{-s} - (p+a)^{-s}$. Then the product (2) has the form

$$\prod_p \frac{1}{1 - y_p(s,a)} \tag{4}$$

d from (3) it follows that

$$\sum_p |y_p(s,a)| < \infty. \tag{5}$$

om Lemma 1, we also know that

$$|y_p(s,a)| < 1. \tag{6}$$

w from (5) and (6) it follows that the product (4) is absolutely convergent.

In fact this convergence is uniform on compact subsets of the s region because one s

$$C_K = \sup_{\substack{p \\ s\in K}} |y_p(a,s)| < 1$$

ere K is a compact subset of the region $\sigma \geqslant \sigma_0$. This completes the proof of Theorem.

Next we are going to consider the asymptotic behaviour of $\zeta_a(1)$ as $a \to \infty$.

eorem 2. *Let* $\sigma > 1$. *Then we have*

$$\lim_{a\to\infty} \zeta_a(s) = \zeta(s).$$

oof. We have

$$\log\zeta_a(s) - \log\zeta(s) = \sum_{p,m\geqslant 1} \frac{1}{m} f_a(m,s)$$

ere

$$f_a(m,s) = \left(\frac{1}{p^s} - \frac{1}{(p+a)^s}\right)^m - \left(\frac{1}{p^s}\right)^m.$$

ence

$$|f_a(m,s)| \leqslant \frac{1}{(p+a)^\sigma}\left(\sum_{j=0}^{m-1}\left(\left|\frac{1}{p^s} - \frac{1}{(p+a)^s}\right|^{m-j-1}\frac{1}{p^{j\sigma}}\right)\right)$$

$$\leqslant \frac{1}{(p+a)^\sigma}\sum_{j=0}^{m-1}\left(\frac{2}{p^\sigma}\right)^{m-j-1} p^{-j\sigma}$$

$$\leqslant \frac{2^{m-1}}{(p+a)^\sigma}\sum_{j=0}^{m-1}\left(\frac{1}{p^\sigma}\right)^{m-1}$$

$$=\frac{2^{m-1}}{(p+a)^\sigma}(mp^{-(m-1)\sigma}).$$

This proves that $\log \zeta_a(s) - \log \zeta(s) \to 0$ as $a \to \infty$.

Theorem 3. *Let $a \to \infty$. Then we have*

$$\zeta_a(1) = \prod_p \frac{1}{\left(1-\frac{1}{p}+\frac{1}{p+a}\right)} = \prod_{p\leqslant a}\frac{1}{1-\frac{1}{p}} + O(1). \tag{7}$$

Proof. Write

$$\prod_p \frac{1}{\left(1-\frac{1}{p}+\frac{1}{p+a}\right)} = \prod_{p\leqslant a}\frac{1}{\left(1-\frac{1}{p}+\frac{1}{p+a}\right)}\prod_{p>a}\frac{1}{\left(1-\frac{1}{p}+\frac{1}{p+a}\right)}$$

$$=\prod\nolimits_1(a)\prod\nolimits_2(a).$$

We will consider $\prod_1(a)$ and $\prod_2(a)$ separately. We have

$$\log\prod\nolimits_2(a) = \sum_{p>a}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{p(p+a)}\right)^m$$

$$<\sum_{p>a}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{a}{p^2}\right)^m < \frac{c_1}{\log a'}$$

because, it is known (by prime number theorem for example) that

$$\sum_{p>a}\frac{1}{p^2} < \frac{c}{a\log a}.$$

So, we have

$$\prod\nolimits_2(a) < \exp\frac{c_2}{\log a}. \tag{8}$$

Then let us consider the asymptotic behaviour of the function

$$g(a) = \prod\nolimits_1(a)\cdot\prod_{p\leqslant a}\left(1-\frac{1}{p}\right) = \prod_{p\leqslant a}\frac{1}{1+\frac{p}{(p-1)(p+a)}}.$$

One has

$$|\log g(a)| = \left|\sum_{\substack{p\leqslant a\\ m\geqslant 1}}(-1)^{m-1}\frac{1}{m}\left(\frac{p}{(p-1)(p+a)}\right)^m\right.$$

$$\leqslant \sum_{\substack{p\leqslant a\\ m\geqslant 1}} \left(\frac{p}{(p-1)(p+a)}\right)^m \leqslant \sum_{\substack{p\leqslant a\\ m\geqslant 1}} \cdot \left(\frac{2}{a}\right)^m$$

$$= \sum_{p\leqslant a} \frac{2}{a} \frac{1}{\left(1-\frac{2}{a}\right)} \leqslant \frac{C_3}{a} \sum_{p\leqslant a} 1 \leqslant \frac{C_4}{\log a}$$

because it is known [1] that

$$\sum_{p\leqslant a} 1 \equiv \pi(a) \sim \frac{a}{\log a}. \tag{9}$$

Therefore one gets

$$g(a) \leqslant \exp \frac{C_4}{\log a}. \tag{10}$$

Now we have

$$\zeta_a(1) = \prod_2 (a) g(a) \cdot \prod_{p\leqslant a} \left(1 - \frac{1}{p}\right)^{-1}$$

and from (8) and (10)

$$\zeta_a(1) = \prod_{p\leqslant a} \frac{1}{1-\frac{1}{p}} + 0(1)$$

since

$$\prod_{p\leqslant a} \frac{1}{1-\frac{1}{p}} \sim e^{\gamma} \log a$$

where γ is the Euler constant. This proves Theorem 3.

3. Discussion

Theorem 3 relates to the asymptotics of the functions $\zeta_a(s)$ and $f_a(s) = \prod_{p\leqslant a} (1-p^{-s})^{-1}$ for $s=1$. In fact there is a relation between the asymptotics of these two functions for any fixed $s \neq 1, 0 < \sigma \leqslant 1$. We consider s real and $0 < s < 1$ for simplicity. We have

$$\log f_a(s) \sim \sum_{p\leqslant a} \frac{1}{p^s}.$$

Then we use the Abel identity

$$\sum_{p\leqslant a} \frac{1}{p^s} = \frac{\pi(a)}{a^s} + s \int_2^a \pi(t) \frac{dt}{t^{1+s}} \tag{11}$$

which is true for any $s\in\mathbb{C}$. Here $\pi(a)$ is the number of prime numbers not exceeding a. By (9) we have for $a\to\infty$

$$\int_2^a \pi(t)\frac{\mathrm{d}t}{t^{1+s}} \sim \int_2^a \frac{\mathrm{d}t}{t^s \log t} \sim \frac{a^{1-s}}{(1-s)\log a}$$

and from (11)

$$\sum_{p\leqslant a}\frac{1}{p^s} \sim \frac{1}{1-s}\frac{a^{1-s}}{\log a} \tag{12}$$

Therefore we have the following

Theorem 4. *Let $0<s<1$. Then we have the following asymptotic equality when $a\to\infty$.*

$$\log\prod_{p\leqslant a}(1-p^{-s})^{-1} \sim \frac{1}{1-s}\frac{a^{1-s}}{\log a}. \tag{13}$$

For the function $\zeta_a(s)$ we have

$$\log\zeta_a(s) \sim \sum_p\left(\frac{1}{p^s}-\frac{1}{(p+a)^s}\right).$$

Now one gets (for $0<s<1$)

$$\sum_p\left(\frac{1}{p^s}-\frac{1}{(p+a)^s}\right) = s\int_2^\infty \pi(t)\left[\frac{1}{t^{s+1}}-\frac{1}{(t+a)^{s+1}}\right]\mathrm{d}t. \tag{14}$$

Since $\pi(x)\sim(x/\log x)$ as $x\to\infty$, we have,

$$\log\zeta_a(s) \sim s\int_2^\infty \frac{t}{\log t}\left(\frac{1}{t^{s+1}}-\frac{1}{(t+a)^{s+1}}\right)\mathrm{d}t.$$

We split the integral into 3 parts say $I_1+I_2+I_3$ where

$$I_1=\int_2^{a\varepsilon}\cdots,\ I_2=\int_{a\varepsilon}^{a\varepsilon^{-1}}\cdots \text{ and } I_3=\int_{a\varepsilon^{-1}}^{\infty}\cdots$$

where $0<\varepsilon<1$ is any fixed constant. It is easy to see that

$$I_1=O\left(\frac{(a\varepsilon)^{1-s}}{\log a}\right) \text{ and } I_3=O\left(a\int_{a\varepsilon^{-1}}^{\infty}\frac{t}{\log t}\frac{(s+1)}{t^{s+2}}\mathrm{d}t\right)=O\left(\frac{a(a\varepsilon^{-1})^{-s}}{\log a}\right)$$

and so I_1 and I_3 are $O\left(\dfrac{a^{1-s}}{\log a}\right)$ as $\varepsilon\to 0$. Now

$$I_2 \sim \int_{a\varepsilon}^{a\varepsilon^{-1}}\frac{t}{\log a}\left(\frac{1}{t^{s+1}}-\frac{1}{(t+a)^{s+1}}\right)\mathrm{d}t=\frac{a^{1-s}}{\log a}\int_\varepsilon^{\varepsilon^{-1}} t\left(\frac{1}{t^{s+1}}-\frac{1}{(t+1)^{s+1}}\right)\mathrm{d}t$$

as can be seen by the substitution $t=au$. Thus we end up with

$$I_2 \sim \frac{a^{1-s}}{\log a} \int_0^\infty t\left(\frac{1}{t^{s+1}} - \frac{1}{(t+1)^{s+1}}\right) dt.$$

It is not hard to prove that the last integral is asymptotic to $[s(1-s)]^{-1}$ $(a^{1-s}/\log a)$ so that we end up with an asymptotic relation which we state as a theorem.

Theorem 5. *For fixed s in* $0 < s < 1$ *we have as* $a \to \infty$,

$$\log f_a(s) \sim \frac{a^{1-s}}{(1-s)\log a}.$$

Remark. It is not very difficult to extend this for complex s with real part lying in the interval $(0, 1]$.

For finding the leading terms in asymptotic expansion for the functions

$$\prod_{p \leqslant a} \frac{1-p^{s-1}}{1-p^{-s}} \text{ and } \prod_p \frac{1-p^{s-1}+(p+a)^{s-1}}{1-p^{-s}+(p+a)^{-s}}$$

we can use above mentioned formulas. It would be interesting to find also the next terms in these expansions. For $0 < s < 1$ one has

$$\log \prod_{p \leqslant a} \frac{1-p^{s-1}}{1-p^{-s}} \sim \left[\frac{1}{1-s}\frac{a^{1-s}}{\log a} - \frac{1}{s}\frac{a^s}{\log a}\right]$$

but in fact we need next terms in the asymptotic expansion.

Note also that the analogous analysis can be applied to the L-function of the form

$$L_a(s, \chi) = \prod_p \frac{1}{1-\chi(p)[p^{-s}-(p+a)^{-s}]} \tag{15}$$

where $\chi(p)$ is the Dirichlet character. To the product (15) the proof of the theorem 1 can be applied and we have also

$$\lim_{a \to \infty} L_a(s, \chi) = L(s, \chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$$

for $\sigma > 1$. It would be interesting to investigate the behaviour $L_a(1, \chi)$. The natural hypothesis is that

$$\lim_{a \to \infty} L_a(1, \chi) = L(1, \chi) = \prod_p \frac{1}{1-\chi(p)p^{-1}}$$

for non principal characters.

It would be interesting to investigate also the questions about the analytical continuation of limit as $a \to \infty$ of $\zeta_a(s)$ in right half plane $\operatorname{Re} s > 0$ and the question about functional equation. Such problems are also interesting for the function

$$\zeta(a, s) = \prod_p \frac{1}{1-(p+a)^{-s}}$$

which is a multiplication analogue of the Hurwitz zeta-function

$$\zeta(a,s)=\sum_{n=1}^{\infty}\frac{1}{(n+a)^s}.$$

References

[1] Titchmarsh E C, *The theory of the Riemann zeta-function*, (Oxford: Clarendon Press) (1986)
[2] Aref́eva I Ya, Dragovič B, Volovich I V, *On the Adelic string Amplitudes*, Preprint, Institute of Physics, Beograd (1988)

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Generalized absolute Cesàro summability of a Fourier series

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Abstract. In an attempt to study the scope of a theorem due to Pati, the authors have established that $\phi(t)\log K|t \in B_u V$ in $(0,\pi) \Rightarrow \sum A_n(x)$ is $|C,0,\beta|$ for $\beta > 1$, at the point $t = x$.

Keywords. Absolute Cesàro summability; Fourier series.

1. Introduction

Let for

$$0 < \delta < 1, \tag{1.1}$$

$\{A_n^{\delta,\beta}\}$ be a sequence of constants defined by

$$\frac{1}{(1-z)^{\delta+1}}\left(\log\frac{a}{1-z}\right)^{\beta} = \sum_{n=0}^{\infty} A_n^{\delta,\beta} z^n, \tag{1.2}$$

for small z, where δ, β are real numbers and, $\delta > -1$ and $a \geqslant 2$.

It is known [7] that for large n,

$$A_n^{\delta,\beta} \sim \frac{n^{\delta}}{\Gamma(\delta+1)}(\log n)^{\beta}, \quad \text{for } \delta \neq -1, -2, \ldots \tag{1.3}$$

$$A_n^{\delta,\beta} \sim (-1)^{\delta-1}(|\delta|-1)\beta n^{\delta}(\log n)^{\beta-1}, \quad \text{for } \delta = -1, -2, \ldots \tag{1.4}$$

Let $\sum a_n$ be an infinite series and $\{s_n\}$ the sequence of its partial sums.

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{A_n^{\delta,\beta}} \sum_{k=0}^{n} A_{n-k}^{\delta-1} s_k \tag{1.5}$$

defines the (C,δ,β) mean of the series Σa_n. If $\lim_{n\to\infty} \sigma_n \to s$, then series $\sum a_n$ is said to be (C,δ,β) summable to s.

The series $\sum a_n$ is said to be absolutely (C,δ,β) summable or summable $|C,\delta,\beta|$ if

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty. \tag{1.6}$$

It is easily seen that $|C,\delta,\beta|$ summability is a regular method for all finite β.

Also for $\beta=0$, the $|C,\delta,\beta|$ summability is the same as $|C,\delta|$ and for $\beta>0$, $|C,\delta|\Rightarrow|C,\delta,\beta|$. For $\beta<0$, $|C,\delta,-\beta|\Rightarrow|C,\delta|$. Thus $|C,\delta,\beta|$ may be regarded as a generalization of absolute Cesàro summability. Further, $|C,0,1|$ is equivalent to harmonic summability.

2. Notations

Let $f(t)$ be a periodic function with period 2π and integrable-L over $(-\pi,\pi)$. We use the following notations:

$$\Phi(t)=\tfrac{1}{2}\{f(x+t)+f(x-t)\},$$

$$g(n,t)=\frac{2}{\pi}\frac{1}{A_n^{0,\beta}}\sum_{v=0}^{n} A_{n-v}^{-1,\beta}\sin vt,$$

$$h(t)=\Phi(t)\log k/t.$$

$[x]$ denotes the greatest integer contained in x. And, for any sequence $\{u_n\}$,

$$\Delta u_n=u_n-u_{n+1}.$$

3. Theorem

In 1936, Bosanquet [1] proved the following theorem for the absolute Cesàro summability of a Fourier series.

Theorem. *If $\Phi(t)\in B\cdot V(0,\pi)$, then the Fourier series of $f(t)$ is summable $|C,\delta|$, at the point $t=x$, for $\delta>0$.*

In 1961, Pati [5] showed that $\Phi(t)\log k/t\in B\cdot V(0,\pi)$ dees not ensure absolute harmonic summability of the Fourier series. His result was:

Theorem. *There exists a function $f(t)$ of class-L such that $\Phi(t)\log k/t$ is a function of bounded variation but its Fourier series at $t=x$ is not summable $|N,1/(n+1)|$.*

Subsequently, Mohanty and Ray [2] considered a more general case by proving the following theorem.

Theorem. *There exists a function $f(t)$ of class-L such that $\Phi(t)\chi(t)\in B\cdot V(0,\pi)$, where $\chi(t)\to+\infty$ as $t\to 0$, but the Fourier series of $f(t)$, at $t=x$, is non-summable $|N,1/(n+1)|$.*

On positive side, Varshney [6] obtained:

Theorem. *If $\Phi(t)\log k/t\in B\cdot V(0,\pi)$ then the Fourier series of $f(t)$, at $t=x$, is summable $|C,0,2|$.*

Generalizing the above theorem we prove:

Theorem. *If $\Phi(t)\log k/t\in B\cdot V$ in $(0,\pi)$ then the Fourier series of $f(t)$ is summable $|C,0,\beta|$ for $\beta>1$ at the point $t=x$.*

We need the following lemmas for the proof of our theorem.

4. Lemmas

Lemma 1. (Nandkishore and Hotta, [3]) *If $\{p_n\}$ is a non-negative and non-increasing sequence, then for $0 \leqslant a < b \leqslant \infty$, $0 \leqslant t \leqslant \pi$ and for any n and a we have*

$$\left|\sum_{k=a}^{b} p_k \exp\left[i(n-k)t\right]\right| \leqslant AP_{\Gamma},$$

where A is an absolute constant and $\mathscr{T} = [1/t]$.

Lemma 2. (Nandkishore and Rath [4])

$$\int_0^t \frac{\cos nu}{\log(k/u)} du = (\log k/t)^{-1} \frac{\sin nt}{n} + 0\left(\frac{1}{n(\log n)^2}\right).$$

Lemma 3.

If

$$g(n,t) = \frac{2}{\pi}\frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta} \sin \nu t$$

then

$$|g(n,t)| = \begin{cases} 0(1), & \textit{for all } t, \\ 0\left(\dfrac{\log \tau}{\log n}\right)^{\beta}, & \textit{for } t > 1/n. \end{cases}$$

Proof.

Since

$$|g(n,t)| = \left|\frac{2}{\pi}\frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta} \sin \nu t\right|,$$

we have

$$\begin{aligned} |g(n,t)| &= \frac{2}{\pi}\left|\frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta} \sin \nu t\right| \\ &= \frac{2}{\pi}\left|\frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{\nu}^{-1,\beta} \sin (n-\nu) t\right| \\ &\leqslant \frac{1}{|A_n^{0,\beta}|} \sum_{\nu=0}^{n} |A_{\nu}^{-1,\beta}| \\ &\quad \text{since } |\sin(n-\nu)t| \leqslant 1 \\ &= 0(1), \quad \text{for all } t. \end{aligned}$$

For $t > 1/n$

$$\begin{aligned} |g(n,t)| &= \frac{2}{\pi}\left|\frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta} \sin \nu t\right| \\ &= \frac{2}{\pi}\left|\frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{\nu}^{-1,\beta} \sin (n-\nu) t\right| \end{aligned}$$

$$\leqslant \frac{1}{A_n^{0,\beta}} 0(A_{\mathscr{T}}^{0,\beta}), \quad \text{by lemma 1}$$
$$= 0\left(\frac{\log \mathscr{T}}{\log n}\right)^{\beta}.$$

5. Proof of the theorem

To prove that the Fourier series of $f(t)$ is summable $|C, 0, \beta|$ at the point $t = x$, that is, it is enough to prove that the series

$$\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is summable $|C, 0, \beta|$, it is enough to prove that the series

$$\sum_{n=1}^{\infty} |\sigma_n - \sigma_{n-1}| < \infty,$$

where

$$\sigma_n = \frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta} s_\nu$$

and

$$s_\nu = A_1 + A_2 + \cdots + A_\nu.$$

Since from the identity

$$n(\sigma_n - \sigma_{n-1}) = \mathscr{T}_n$$

where

$$\mathscr{T}_n = \frac{1}{A_n^{0,\beta}} \sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta} \nu A_\nu,$$

and $\mathscr{T}_n$ denotes the $(C, 0, \beta)$ mean of the sequence $\{nA_n\}$.

Hence our theorem is established if we prove

$$\sum_{n=1}^{\infty} \frac{|\mathscr{T}_n|}{n} < \infty.$$

Now, since

$$A_n = \frac{2}{\pi} \int_0^{\pi} \Phi(t) \cos nt \, dt$$
$$= \frac{2}{\pi} \int_0^{\pi} \Phi(t) \log k/t \frac{\cos nt}{\log k/t} \, dt$$
$$= \frac{2}{\pi} \int_0^{\pi} h(t) \frac{\cos nt}{\log k/t} \, dt$$

where $h(t) = \Phi(t) \log k/t$.

Thus

$$A_n = \frac{2}{\pi}\left[h(t) \int_0^t \frac{\cos nu}{\log k/t}\, dt \right]_0^\pi - \frac{2}{\pi}\int_0^\pi h(t) \int_0^t \frac{\cos nu}{\log k/u}\, du$$

$$= \frac{2}{\pi} h(\pi) \int_0^\pi \frac{\cos nu}{\log k/u}\, du - \frac{2}{\pi}\int_0^\pi dh(t) \int_0^t \frac{\cos nu}{\log k/u}\, du.$$

Now, by Lemma 2

$$\int_0^t \frac{\cos nu}{\log k/u}\, du = (\log k/t)^{-1}\frac{\sin nt}{n} + 0\left(\frac{1}{n(\log n)^2}\right).$$

Therefore

$$A_n = 0\left(\frac{1}{n(\log n)^2}\right) - \frac{2}{\pi}\int_0^\pi dh(t)\left[(\log k/t)^{-1}\frac{\sin nt}{n} + 0\left(\frac{1}{n(\log n)^2}\right)\right]$$

$$= 0\left(\frac{1}{n(\log n)^2}\right) - \frac{2}{\pi}\int_0^\pi dh(t)(\log k/t)^{-1}\frac{\sin nt}{n}.$$

So

$$\mathscr{T}_n = \frac{1}{A_n^{0,\beta}}\sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta}\nu A_\nu$$

$$= \frac{1}{A_n^{0,\beta}}\sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta}\left[0\left(\frac{\nu}{\nu(\log \nu)^2}\right) - \frac{2\nu}{\pi}\int_0^\pi dh(t)(\log k/t)^{-1}\frac{\sin \nu t}{\nu}\right]$$

$$= 0\left(\frac{1}{(\log n)^2}\right)\frac{1}{A_n^{0,\beta}}\cdot\sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta}$$

$$- \int_0^\pi dh(t)(\log k/t)^{-1}\left[\frac{2}{\pi}\frac{1}{A_n^{0,\beta}}\sum_{\nu=0}^{n} A_{n-\nu}^{-1,\beta}\sin \nu t\right]$$

$$= 0\left(\frac{1}{(\log n)^2}\right) - \int_0^\pi dh(t)(\log k/t)^{-1} g(n,t).$$

Hence our theorem is established if we prove

$$\sum_{n=1}^{\infty}\frac{|\mathscr{T}_n|}{n} < \infty.$$

Now

$$\sum_{n=1}^{\infty}\frac{|\mathscr{T} n|}{n} = \sum_{n=1}^{\infty}\frac{1}{n}\left|0\left(\frac{1}{(\log n)^2}\right) - \int_0^\pi dh(t)(\log k/t)^{-1} g(n,t)\right|$$

$$\leqslant \sum_{n=1}^{\infty}\frac{1}{n(\log n)^2} + \int_0^\pi |dh(t)|(\log k/t)^{-1}\sum_{n=1}^{\infty}\frac{1}{n}|g(n,t)|$$

$$= \Sigma_1 + \Sigma_2, \text{ say.}$$

Since the series $\sum_{n=1}^{\infty}(1/n(\log n)^2)$ is absolutely convergent

Since $h(t) \in B \cdot V(0, \pi)$ our theorem is proved if we can show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |g(n, t)| = 0(\log \mathscr{T})$$

Now

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} |g(n, t)| &= \sum_{n=1}^{\mathscr{T}} \frac{1}{n} |g(n, t)| + \sum_{n=\mathscr{T}+1}^{\infty} \frac{1}{n} |g(n, t)| \\
&= 0(1) \sum_{n=1}^{\mathscr{T}} \frac{1}{n} + \sum_{n=\mathscr{T}+1}^{\infty} \frac{1}{n} 0\left(\frac{\log \mathscr{T}}{\log n}\right)^{\beta} \text{ by Lemma 3} \\
&= 0(\log \mathscr{T}) + 0(\log \mathscr{T})^{\beta} \sum_{n=\mathscr{T}+1}^{\infty} \frac{1}{n(\log n)^{\beta}} \text{ where } \beta > 1 \\
&= 0(\log \mathscr{T}) + 0(\log \mathscr{T})^{\beta} \frac{(\log \mathscr{T})}{(\log \mathscr{T})^{\beta}} \\
&= 0(\log \mathscr{T}) + 0(\log \mathscr{T}) \\
&= 0(\log \mathscr{T})
\end{aligned}$$

This completes the proof of the theorem.

References

[1] Bosanquet L S, The absolute Cesàro summability of a Fourier series, *Proc. London Math. Soc.* (2), **41** (1936) 517–528

[2] Mohanty R and Ray B K, On the non-absolute summability of a Fourier series and the conjugate series by a Nörlund method, *Proc. Camb. Philos. Soc.* **63** (1967) 407–411

[3] Nandkishore and Hotta G C, On absolute matrix summability of a Fourier series, *Indian J. Math.* **13** (1971) 90–110

[4] Nandkishore and Rath S, A note on the absolute Nörlund summability of a Fourier series and allied series, *The Aligarh Bull. Math.* (1980) and Ph.D. thesis, Berhampur Univ. 139

[5] Pati T, The non-absolute summability of a Fourier series by a Nörlund method, *J. Indian Math. Soc.* **25** (1961) 197–214

[6] Varshney O P, On the absolute summability of a Fourier series by a Nörlund method, *Rourkee Residential J.* **6** (1963) 103–113

[7] Zygmund A, *Trigonometric series* (Cambridge: University Press) Vol. I (1959)

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Fractional integral formulae involving a general class of polynomials and the multivariable H-function

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Abstract. We obtain two fractional integral formulae involving a general class of polynomials and the multivariable H-function. On account of the most general nature of the polynomials and the multivarible H-function involved herein, our findings provide interesting unifications and extensions of a number of (known and new) results. We have mentioned here only two such results.

Keywords. Riemann–Liouville operator; Erdélyi-Kober operator; multivariable H-function; general class of polynomials; fractional integral formulae.

1. Introduction and definitions

The familiar fractional integral operator (FIO) is defined and represented in the present paper as

$$ {}_cI_x^{\nu}\{f(x)\} = \frac{1}{\Gamma(\nu)} \int_c^x (x-t)^{\nu-1} f(t)\,\mathrm{d}t, \quad \mathrm{Re}(\nu) > 0 \tag{1.1} $$

The special case of the above operator (when $c = 0$) is well known in the literature as Riemann–Liouville fractional integral operator and is written as $I_x^{\nu}\{f(x)\}$.

Also the FIO investigated by Erdélyi-Kober is defined and represented as [2]

$$ I_x^{\eta,\nu}\{f(x)\} = \frac{x^{-\eta-\nu+1}}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{\eta-1} f(t)\,\mathrm{d}t, \quad \mathrm{Re}(\nu) > 0, \eta > 0 \tag{1.2} $$

which is obviously a generalization of the Riemann–Liouville FIO.

The multivariable H-function has been defined by Srivastava and Panda [8] We shall use the following contracted form [7, p. 251, eq. (C.1)]:

$$ H[z_1,\ldots,z_r] = H^{0,N:M',N';\cdots;M^{(r)},N^{(r)}}_{P,Q:P',Q';\cdots;P^{(r)},Q^{(r)}} $$

$$ \times \left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \middle| \begin{matrix} (a_j;\alpha'_j,\ldots,\alpha_j^{(r)})_{1,P} : (c'_j,\gamma'_j)_{1,P'};\cdots;(c_j^{(r)},\gamma_j^{(r)})_{1,P^{(r)}} \\ (b_j;\beta'_j,\ldots,\beta_j^{(r)})_{1,Q} : (d'_j,\delta'_j)_{1,Q'};\cdots;(d_j^{(r)},\delta_j^{(r)})_{1,Q^{(r)}} \end{matrix} \right] \tag{1.3} $$

to denote the H-function of r complex variables $z_1,\ldots,z_r$. All the Greek letters are

assumed to be positive real numbers for standardization purposes; the definition of the multivariable H-function will, however, be meaningful even if some of these quantities are zero. Various special cases and details of this function can be found in the paper referred to above.

Srivastava has also introduced the general class of polynomials (see [3] and [4]):

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots, \tag{1.4}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n, k \geqslant 0)$ are arbitrary constants, real or complex. By suitably specializing the coefficients $A_{n,k}$, the above general class of polynomials can be reduced to a large spectrum of polynomials as cited in the papers referred to above ([3] and [4]).

2. Results required

The following results will be required in the sequel:

(I)
$$ {}_cI_x^\nu\{x^\lambda\} = \sum_{s=0}^{\infty} \frac{(-1)^s \Gamma(\lambda+1)(x-c)^{s+\nu} x^{\lambda-s}}{\Gamma(\nu)\Gamma(\lambda-s+1)\cdot(s+\nu)\cdot s!}, \quad \mathrm{Re}(\lambda) > -1 \tag{2.1}$$

(II)
$$\sum_{s=0}^{\infty} \frac{(-1)^s}{\Gamma(\nu)\cdot(s+\nu)\cdot s!} H^{0,N\,:\,*}_{P,Q+1\,:\,*}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P} & :* \\ (s-k;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q} & :* \end{matrix}\right]$$

$$= H^{0,N\,:\,*}_{P,Q+1\,:\,*}\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P} & :* \\ (-\nu-k;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q} & :* \end{matrix}\right]. \tag{2.2}$$

the asterisk $(*)$ in (2.2) indicates that the parameters at these places are the same as the parameters of the multivariable H-function in (1.3).

(III)
$$I_x^{\eta,\nu}\{x^\lambda\} = \frac{\Gamma(\eta+\lambda)}{\Gamma(\eta+\lambda+\nu)} x^\lambda, \quad \mathrm{Re}(\lambda) > -\eta. \tag{2.3}$$

The results given above are all consequences of the binomial series

$$(1-t)^\lambda = \sum_{s=0}^{\infty} \frac{(-\lambda)_s t^s}{s!} (|t| < 1)$$

and would follow from it on suitable integrations.

3. The fractional integral formulae

Recently Srivastava and Garg [5, p. 688] have obtained two integrals involving a general class of polynomials and the multivariable H-function. Here we obtain the

following fractional integral formulae involving them:

$$
{}_cI_x^{\nu}\{x^{\rho}(x+b)^{\sigma}S_n^m[ax^u(x+b)^v]H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\}
$$

$$
= b^{\sigma}x^{\rho}\sum_{s,l=0}^{\infty}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}A_{n,k}a^kb^{vk-l}(-1)^s(x-c)^{s+\nu}x^{uk+l-s}}{\Gamma(\nu)\cdot(s+\nu)\cdot s!\,l!\,k!}\right]\cdot H^{0,N+2\ :*}_{P+2,Q+2:*}
$$

$$
\times\left[\begin{matrix} z_1b^{v_1}x^{u_1}\\ \vdots \\ z_rb^{v_r}x^{u_r}\end{matrix}\left|\begin{matrix}(-\sigma-vk;v_1,\ldots,v_r),(-\rho-uk-l;u_1,\ldots,u_r),\ (a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}:*\\ (l-\sigma-vk;v_1,\ldots,v_r),(s-\rho-uk-l;u_1,\ldots,u_r),\ (b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right.\right] \tag{3.1}
$$

provided that

(i) $\mathrm{Re}(\nu)>0;\ \min\{u_i,v_i,u,v\}>0,\quad i=1,\ldots,r;\ \left|\arg\left(\frac{x}{b}\right)\right|<\pi,$

(ii) $\mathrm{Re}(\rho)+\sum_{i=1}^{r}u_i\min_{1\leqslant j\leqslant M^{(i)}}\left\{\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\}>-1,$

(iii) $\Omega_i>0,\ |\arg z_i|<\frac{1}{2}\Omega_i\pi,\quad \forall i\in\{1,2,\ldots,r\},$
where

$$
\Omega_i=-\sum_{j=N+1}^{P}\alpha_j^{(i)}-\sum_{j=1}^{Q}\beta_j^{(i)}+\sum_{j=1}^{N^{(i)}}\gamma_j^{(i)}-\sum_{j=N^{(i)}+1}^{P^{(i)}}\gamma_j^{(i)}+\sum_{j=1}^{M^{(i)}}\delta_j^{(i)}-\sum_{j=M^{(i)}+1}^{Q^{(i)}}\delta_j^{(i)}
$$

(iv) the series occurring on right-hand side of (3.1) is absolutely convergent.

Proof. To establish (3.1), we first express the general class of polynomials occurring on left-hand side of it in the series form given by (1.4) and replace multivariable H-function by its Mellin-Barnes contour integral, collect the powers of x and $(x+b)$ and apply the binomial expansion

$$
(x+b)^{\sigma}=b^{\sigma}\sum_{l=0}^{\infty}\binom{\sigma}{l}\left(\frac{x}{b}\right)^l,\quad \left|\frac{x}{b}\right|<1. \tag{3.2}
$$

Further, making use of the result (2.1) and interpreting the resulting Mellin-Barnes contour integral as the H-function of r variables, we shall arrive at (3.1).

Following (3.1) and using the result (2.3) in place of (2.1), we easily arrive at the following fractional integral formula for Erdélyi-Kober operator defined by (1.2):

$$
I_x^{\eta,\nu}\{x^{\rho}(x+b)^{\sigma}S_n^m[ax^u(x+b)^v]H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\}
$$

$$
= b^{\sigma}x^{\rho}\sum_{l=0}^{\infty}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}A_{n,k}a^kb^{vk-l}x^{uk+l}}{l!\,k!}\right]H^{0,N+2\ :*}_{P+2,Q+2:*}
$$

$$
\left[\begin{matrix} z_1b^{v_1}x^{u_1}\\ \vdots \\ z_rb^{v_r}x^{u_r}\end{matrix}\left|\begin{matrix}(-\sigma-vk;v_1,\ldots,v_r),(1-\eta-\rho-uk-l;u_1,\ldots,u_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}\ :*\\ (l-\sigma-vk;v_1,\ldots,v_r),(-\nu+1-\eta-\rho-uk-l;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right.\right] \tag{3.3}
$$

where $\eta>0$ and

$$
\mathrm{Re}(\rho)+\sum_{i=1}^{r}u_i\min_{1\leqslant j\leqslant M^{(i)}}\left\{\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right\}>-\eta.
$$

The remaining conditions of validity being the same as given in (3.1).

4. Special cases and applications

If we take $c = 0$ in our main integral formula (3.1) and use the result (2.2), we arrive at the following fractional integral formula:

$$I_x^v\{x^\rho(x+b)^\sigma S_n^m[ax^u(x+b)^v]H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\}$$
$$= b^\sigma x^{\rho+v}\sum_{l=0}^{\infty}\sum_{k=0}^{[n/m]}\left[\frac{(-n)_{mk}A_{n,k}a^kb^{vk-l}x^{uk+l}}{l!\,k!}\right]H^{0,N+2\,:*}_{P+2,Q+2:*}$$
$$\times\left[\begin{matrix}z_1b^{v_1}x^{u_1}\\ \vdots\\ z_rb^{v_r}x^{u_r}\end{matrix}\middle|\begin{matrix}(-\sigma-vk;v_1,\ldots,v_r),(-\rho-uk-l;u_1,\ldots,u_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}\ :*\\ (l-\sigma-vk;v_1,\ldots,v_r),(-v-\rho-uk-l;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right] \tag{4.1}$$

The conditions of validity of (4.1) are directly obtainable from that of (3.1).

Again, if we take $n = 0$ (the polynomial $S_0^m[x]$ will reduce to $A_{0,0}$) in (4.1), we get a result which is in essence the same as obtained earlier by Srivastava and Goyal [6, p. 644].

The importance of our main integral formulae lies in their manifold generality. Firstly, in view of the generality of the polynomials $S_n^m[x]$, on specializing the coefficients $A_{n,k}$ and making a free use of the special cases of $S_n^m[x]$ listed in a paper by Srivastava [4], our results can be reduced to a large number of integral formulae involving simpler polynomials.

Secondly, by specializing the various parameters and variables in the multivariable H-function, we can obtain from our results, several integral formulae involving a remarkably wide variety of useful functions (or products of several such functions), which are expressible in terms of E, F, G and H-functions of one and several variables.

We shall give here only one such formula for the sake of illustration. Thus, if we take $a = u = 1$, $v = 0$, $m = 1$ and $A_{n,k} = [\Gamma(1+\alpha+\lambda n)]/[n!\,\Gamma(1+\alpha+\lambda k)]$ in (4.1), the polynomial $S_n^1[x]$ reduces to $z_n^\alpha(x^{1/\lambda};\lambda)$ (known as Konhauser biorthogonal polynomials [1, p. 304] and we easily get

$$I_x^v\{x^\rho(x+b)^\sigma z_n^\alpha(x^{1/\lambda};\lambda)H[z_1x^{u_1}(x+b)^{v_1},\ldots,z_rx^{u_r}(x+b)^{v_r}]\}$$
$$= b^\sigma x^{\rho+v}\sum_{l=0}^{\infty}\sum_{k=0}^{n}\frac{(-n)_k\Gamma(1+\alpha+\lambda n)b^{-l}x^{k+l}}{l!\,k!\,n!\,\Gamma(1+\alpha+\lambda k)}H^{0,N+2\,:*}_{P+2,Q+2:*}$$
$$\times\left[\begin{matrix}z_1b^{v_1}x^{u_1}\\ \vdots\\ z_rb^{v_r}x^{u_r}\end{matrix}\middle|\begin{matrix}(-\sigma;v_1,\ldots,v_r),(-\rho-k-l;u_1,\ldots,u_r),(a_j;\alpha_j',\ldots,\alpha_j^{(r)})_{1,P}\ :*\\ (l-\sigma;v_1,\ldots,v_r),(-v-\rho-k-l;u_1,\ldots,u_r),(b_j;\beta_j',\ldots,\beta_j^{(r)})_{1,Q}:*\end{matrix}\right] \tag{4.2}$$

The conditions of validity of (4.2) are directly obtainable from those of (3.1). Further, on taking $\lambda = 1$ in (4.2), we get the corresponding fractional integral formula involving Laguerre polynomials $L_n^{(\alpha)}(x)$.

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References

[1] Konhauser J D E, Biorthogonal polynomials suggested by the Laguerre polynomials, *Pacific J. Math.* **21** (1967) 303–314

[2] Ross B, *Fractional calculus and its applications, Lecture notes in mathematics* (New York: Springer Verlag) Vol. 457 (1975)

[3] Srivastava H M, A contour integral involving Fox's H-function, *Indian J. Math.* **14** (1972) 1–6

[4] Srivastava H M, The Weyl fractional integral of a general class of polynomials, *Boll. Un. Mat. Ital.* **B(6) 2** (1983) 219–228

[5] Srivastava H M and Garg M, Some integrals involving a general class of polynomials and the multivariable H-function, *Rev. Roumaine Phys.* **32** (1987) 685–692

[6] Srivastava H M and Goyal S P, Fractional derivatives of the H-function of several variables, *J. Math. Anal. Appl.* **112** (1985) 641–651

[7] Srivastava H M, Gupta K C and Goyal S P, *The H-functions of one and two variables with applications* (New Delhi: South Asian Publishers) (1982)

[8] Srivastava H M and Panda R, Some bilateral generating functions for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283–284** (1976) 265–274

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Asymptotic expansions of the Mehler-Fock transform

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Abstract. Asymptotic expansions in the two limits $x \to +\infty$ and $x \to 0+$ are obtained for the Mehler-Fock transform

$$I(x) = \int_0^\infty P^m_{-\frac{1}{2}+i\tau}(\cosh x) h(\tau)\,\mathrm{d}\tau,$$

where $P^m_{-\frac{1}{2}+i\tau}(\cosh x)$ is the associated Legendre function.

Keywords. Mehler-Fock transform; asymptotic expansions; Legendre function.

1. Introduction

Let $h(\tau)$ be a locally integrable function on $[0, \infty)$. The Mehler-Fock transform of $h(\tau)$, when it exists, is defined by

$$I(x) = \int_0^\infty P^m_{-\frac{1}{2}+i\tau}(\cosh x) h(\tau)\,\mathrm{d}\tau, \quad \mathrm{Re}\, m > -\tfrac{1}{2}, \tag{1.1}$$

where $P^m_{-\frac{1}{2}+i\tau}(\cosh x)$ is the associated Legendre function [1, p. 156]. This transform plays an important role in a number of problems concerning diffraction of waves by wedges. Here the integration is with respect to the order of the Legendre function and hence the evaluation of transform is considerably difficult. This suggests that it may be useful to make a systematic investigation of the behaviour of this transform for both large and small values of the variable x. As far as we are aware, there is no asymptotic result available in the literature for this transform.

For the large x-behaviour of $I(x)$, we shall assume that $h*(\tau) = \cosh \pi\tau\, h(\tau)$ is either exponentially decaying, i.e.

$$h^*(\tau) = O(\exp(-\nu\tau)) \quad \text{as } \tau \to +\infty \tag{1.2}$$

for some $\nu > 0$, or has an asymptotic expansion of the form

$$h^*(\tau) \sim \exp(ic\tau) \sum_{s=0}^{\infty} a_s \tau^{-s-\alpha} \quad \text{as } \tau \to +\infty, \tag{1.3}$$

where $0 < \alpha \leqslant 1$ and c is a real number.

To obtain the small-x behaviour of $I(x)$, we shall impose the condition that $h(\tau)$ satisfies the behaviour

$$h(\tau) \sim \sum_{s=0}^{\infty} b_s \tau^{s+\lambda-1} \quad \text{as } \tau \to 0+, \tag{1.4}$$

where $0 < \lambda \leqslant 1$.

2. Exponentially decaying or oscillatory $h^*(\tau)$

We are first concerned with the behaviour of $I(x)$ as $x \to +\infty$. Thus we assume that either (1.2) or (1.3) holds. In addition, we assume that $h(\tau)$ is locally absolutely integrable on $[0, \infty)$ and as $\tau \to 0+$

$$h(\tau) = O(\tau^b), \quad b > -1. \tag{2.1}$$

We now return to (1.1). The kernel function $P^m_{-\frac{1}{2}+i\tau}(\cosh x)$ has the well known integral representation

$$P^m_{-\frac{1}{2}+i\tau}(\cosh x) = (-1)^m 2^{\frac{1}{2}} \pi^{-\frac{3}{2}} \cosh(\pi\tau)(\sinh x)^m \int_0^\infty t^{m-\frac{1}{2}} \exp(-t\cosh x) K_{i\tau}(t)\, dt. \tag{2.2}$$

Moreover

$$\int_0^\infty t^{m-\frac{1}{2}} \exp(-t\cosh x) K_{i\tau}(t)\, dt = \Gamma(m+\tfrac{1}{2}) \int_0^\infty \frac{\cos(\tau u)\, du}{(\cosh u + \cosh x)^{m+\frac{1}{2}}}. \tag{2.3}$$

Substituting (2.3) in (2.2) we have

$$P^m_{-\frac{1}{2}+i\tau}(\cosh x) = (-1)^m 2^{\frac{1}{2}} \pi^{-\frac{3}{2}} \cosh(\pi\tau)(\sinh x)^m \Gamma(m+\tfrac{1}{2}) \int_0^\infty \frac{\cos(\tau u)\, du}{(\cosh u + \cosh x)^{m+\frac{1}{2}}} \tag{2.4}$$

[2, p. 387]. Substituting (2.4) in (1.1) and reversing the order of integration we have

$$\begin{aligned} I(x) &= (-1)^m 2^{\frac{1}{2}} \pi^{-\frac{3}{2}} \Gamma(m+\tfrac{1}{2})(\sinh x)^m \\ &\quad \times \int_0^\infty \cosh(\pi\tau) h(\tau)\, d\tau \int_0^\infty \frac{\cos(\tau u)\, du}{(\cosh u + \cosh x)^{m+\frac{1}{2}}} \\ &= (-1)^m 2^{\frac{1}{2}} \pi^{-\frac{3}{2}} \Gamma(m+\tfrac{1}{2})(\sinh x)^m \int_0^\infty \frac{du}{(\cosh u + \cosh x)^{m+\frac{1}{2}}} \\ &\quad \times \int_0^\infty \cosh(\pi\tau) h(\tau) \cos(\tau u)\, d\tau \\ &= (-1)^m 2^{\frac{1}{2}} \pi^{-\frac{3}{2}} \Gamma(m+\tfrac{1}{2})(\sinh x)^m \int_0^\infty \frac{H_c(u)\, du}{(\cosh u + \cosh x)^{m+\frac{1}{2}}} \end{aligned} \tag{2.5}$$

where $H_c(u)$ is the Fourier-cosine transform of $h^*(\tau)$ i.e.

$$H_c(u) = \int_0^\infty h^*(\tau)\cos(\tau u)\,d\tau \tag{2.6}$$

where

$$h^*(\tau) = \cosh(\pi\tau)h(\tau). \tag{2.7}$$

The change in the order of integration is justified by absolute convergence when (1.2) holds and by uniform convergence when (1.3) holds.

If $h^*(\tau)$ satisfies (1.2), then the moments

$$\mu_n = \int_0^\infty \tau^n h^*(\tau)\,d\tau, \quad n = 0, 1, 2\ldots \tag{2.8}$$

exist and it is easy to see that

$$H_c(u) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\mu_{2n}u^{2n}. \tag{2.9}$$

If $h^*(\tau)$ satisfies (1.3) with $c \neq 0$, then the generalized Mellin transform defined by

$$M[h^*; z] = \int_0^1 \tau^{z-1}h^*(\tau)\,d\tau + \int_1^\infty \tau^{z-1}h^*(\tau)\,d\tau \tag{2.10}$$

can be analytically continued to the right half plane $\operatorname{Re} z > -b$; [3; Lemma 3]. Furthermore, from Theorem 3 in [3] with $f(\tau) = \cos\tau$ and $\lambda = 1/u$, we have

$$H_c(u) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} M[h^*(\tau); 2n+1]u^{2n}. \tag{2.11}$$

Substituting (2.11) in (2.5), we have

$$\begin{aligned} I(x) &= (-1)^m 2^{\frac{1}{2}}\pi^{-\frac{3}{2}}\Gamma(m+\tfrac{1}{2})(\sinh x)^m \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!} \\ &\quad\times M[h^*(\tau); 2n+1]\int_0^\infty u^{2n}(\cosh u + \cosh x)^{-m-\frac{1}{2}}\,du \\ &= (-1)^m 2^{\frac{1}{2}}\pi^{-\frac{3}{2}}\Gamma(m+\tfrac{1}{2})(\sinh x)^m \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!} M[h^*(\tau); 2n+1] I_m^n(x) \end{aligned} \tag{2.12}$$

where

$$I_m^n(x) = \int_0^\infty u^{2n}(\cosh u + \cosh x)^{-m-\frac{1}{2}}\,du. \tag{2.13}$$

Now, we find the asymptotic expansion of $I_m^n(x)$, for $x \to \infty$,

$$\begin{aligned} I_m^n(x) &= \int_0^\infty u^{2n}(\cosh u + \cosh x)^{-m-\frac{1}{2}}\,du \\ &= (\cosh x)^{-m-\frac{1}{2}}\int_0^x u^{2n}\sum_{r=0}^{\infty}\binom{-m-\frac{1}{2}}{r}\frac{(\cosh u)^r}{(\cosh x)^r}\,du \end{aligned}$$

$$+\int_x^\infty u^{2n}(\cosh u)^{-m-\frac{1}{2}}\sum_{r=0}^\infty \binom{-m-\frac{1}{2}}{r}\frac{(\cosh x)^r}{(\cosh u)^r}\,du$$

$$=(\cosh x)^{-m-\frac{1}{2}}\sum_{r=0}^\infty \binom{-m-\frac{1}{2}}{r}\cosh^{-r}x\int_0^x u^{2n}\cosh^r u\,du$$

$$+\sum_{r=0}^\infty \binom{-m-\frac{1}{2}}{r}\cosh^r x\int_x^\infty u^{2n}\cosh^{-m-\frac{1}{2}-r}u\,du$$

$$=\cosh^{-m-\frac{1}{2}}x\sum_{r=0}^\infty \binom{-m-\frac{1}{2}}{r}\cosh^{-r}x a_r(x)$$

$$+\sum_{r=0}^\infty \binom{-m-\frac{1}{2}}{r}\cosh^r x b_r(x), \tag{2.14}$$

where

$$a_r(x)=\int_0^x u^{2n}\cosh^r u\,du$$

and

$$b_r(x)=\int_x^\infty u^{2n}\cosh^{-m-\frac{1}{2}-r}u\,du.$$

We find that

$$a_0(x)=\frac{x^{2n+1}}{2n+1},$$

$$a_1(x)=x^{2n}\sinh x-2nx^{2n-1}\cosh x+2n(2n-1)x^{2n-2}\sinh x+\cdots$$
$$+(2n)!\sinh x,$$

$$a_2(x)=\frac{x^{2n+1}}{2n+1}+\frac{x^{2n}\sinh 2x}{2^2}-\frac{2nx^{2n-1}\cosh 2x}{2^3}+\frac{2n(2n-1)x^{2n-2}\sinh 2x}{2^4}$$
$$+\cdots+\frac{(2n)!\sinh 2x}{2^{2n+2}}$$

and so on. Also,

$$b_0(x)=2^{m+\frac{1}{2}}\sum_{q=0}^\infty \frac{(-1)^q(m+\frac{1}{2})_q}{(q)!}\frac{\Gamma(2n+1;(m+\frac{1}{2}+2q)x)}{\{(m+\frac{1}{2})+2q\}^{2n+1}},$$

$$b_1(x)=2^{m+\frac{3}{2}}\sum_{q=0}^\infty \frac{(-1)^q(m+\frac{3}{2})_q}{(q)!}\frac{\Gamma(2n+1;(m+\frac{3}{2}+2q)x)}{\{(m+\frac{3}{2})+2q\}^{2n+1}}$$

$$b_2(x)=2^{m+\frac{5}{2}}\sum_{q=0}^\infty \frac{(-1)^q(m+\frac{5}{2})_q}{(q)!}\frac{\Gamma(2n+1;(m+\frac{5}{2}+2q)x)}{\{(m+\frac{5}{2})+2q\}^{2n+1}}$$

and so on. Here $\Gamma(\alpha;y)$ is the incomplete gamma function defined by

$$\Gamma(\alpha;y)=\int_y^\infty t^{\alpha-1}\exp(-t)\,dt,\quad y>0.$$

Substituting (2.14) in (2.12), we have the asymptotic expansion of $I(x)$.

3. Asymptotic expansion for small value of x

We find the asymptotic expansion of (2.13) for small value of x i.e. $x \to 0+$. Since

$$(\cosh u + \cosh x)^{-m-\frac{1}{2}} = (1+\cosh u)^{-m-\frac{1}{2}} + \frac{(-m-\frac{1}{2})}{2!}(1+\cosh u)^{-m-\frac{3}{2}}x^2 + \cdots \tag{3.1}$$

Therefore

$$I_m^n(x) = \int_0^\infty u^{2n}\left[(1+\cosh u)^{-m-\frac{1}{2}} - \frac{(m+\frac{1}{2})}{2!}(1+\cosh u)^{-m-\frac{3}{2}}x^2 + \cdots\right]du. \tag{3.2}$$

Now

$$\begin{aligned}
&\int_0^\infty u^{2n}(1+\cosh u)^{-m-\frac{1}{2}}\,du \\
&= 2^{-m-\frac{1}{2}}\int_0^\infty \left(\frac{\exp(u/2)+\exp(-u/2)}{2}\right)^{-2(m+\frac{1}{2})} u^{2n}\,du \\
&= 2^{m+\frac{1}{2}}\int_0^\infty u^{2n}\exp(-(m+\tfrac{1}{2})u)\{1+\exp(-u)\}^{-(m+\frac{1}{2})2}\,du \\
&= 2^{m+\frac{1}{2}}\int_0^\infty u^{2n}\exp(-(m+\tfrac{1}{2})u)\,{}_1F_0(2(m+\tfrac{1}{2}); -\exp(-u))\,du \\
&= 2^{m+\frac{1}{2}}\int_0^\infty u^{2n}\exp(-(m+\tfrac{1}{2})u)\sum_{r=0}^\infty \frac{(-1)^r(2(m+\frac{1}{2}))_r}{(r)!}\exp(-ur)\,du \\
&= 2^{m+\frac{1}{2}}\sum_{r=0}^\infty \frac{(-1)^r(2(m+\frac{1}{2}))_r}{(r)!}\int_0^\infty u^{2n}\exp(-(m+\tfrac{1}{2})u-ur)\,du \\
&= 2^{m+\frac{1}{2}}\sum_{r=0}^\infty \frac{(-1)^r(2(m+\frac{1}{2}))_r}{(r)!}\,\frac{\Gamma(2n+1)}{\{(m+\frac{1}{2})+r\}^{2n+1}}.
\end{aligned} \tag{3.3}$$

Similarly

$$\begin{aligned}
&\int_0^\infty u^{2n}(1+\cosh u)^{-m-\frac{3}{2}}\,du \\
&= 2^{m+\frac{3}{2}}\left(\sum_{r=0}^\infty \frac{(-1)^r(2(m+\frac{3}{2}))_r}{(r)!}\int_0^\infty u^{2n}\exp[(-m+\tfrac{3}{2})u-ur)]\,du\right. \\
&= 2^{m+\frac{3}{2}}\sum_{r=0}^\infty \frac{(-1)^r(2(m+\frac{3}{2}))_r}{(r)!}\,\frac{\Gamma(2n+1)}{\{(m+\frac{3}{2})+r\}^{2n+1}}
\end{aligned} \tag{3.4}$$

and so on.

Substituting (3.3) and (3.4) in (3.2) we have

$$\begin{aligned}
I_m^n(x) &= 2^{m+\frac{1}{2}}\sum_{r=0}^\infty \frac{(-1)^r(2(m+\frac{1}{2}))_r}{(r)!}\,\frac{\Gamma(2n+1)}{\{(m+\frac{1}{2})+r\}^{2n+1}} \\
&\quad - \frac{2^{m+\frac{3}{2}}(m+\frac{1}{2})}{2!}\sum_{r=0}^\infty \frac{(-1)^r(2(m+\frac{3}{2}))_r}{(r)!}\,\frac{\Gamma(2n+1)}{\{(m+\frac{3}{2})+r\}^{2n+1}}x^2 + \cdots.
\end{aligned}$$

Thus

$$I_m^n(x) = \sum_{s=0}^{\infty} (-1)^s C_s x^{2s}, \tag{3.5}$$

where

$$C_0 = 2^{m+\frac{1}{2}} \sum_{r=0}^{\infty} \frac{(-1)^r (2(m+\frac{1}{2}))_r}{r!} \frac{\Gamma(2n+1)}{\{(m+\frac{1}{2})+r\}^{2n+1}},$$

$$C_1 = \frac{2^{m+\frac{3}{2}}(m+\frac{1}{2})}{2!} \sum_{r=0}^{\infty} \frac{(-1)^r (2(m+\frac{3}{2}))_r}{r!} \frac{\Gamma(2n+1)}{\{(m+\frac{3}{2})+r\}^{2n+1}}$$

and so on.

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References

[1] Erdélyi A, Magnus W, Oberhettinger F and Tricomi F, *Higher transcendental functions* Vol 1 (New York: McGraw-Hill) (1953)

[2] Sneddon I N, *The use of integral transforms* (New York: McGraw Hill) (1972)

[3] Wong R, Explicit error terms for asymptotic expansions of Mellin convolution, *J. Math. Anal. Appl.* **72** (1979) 740–756

[4] Wong R, Asymptotic expansions of the Kontorovich-Lebedev transform, *Appl. Anal.* **12** (1981) 161–172

ian Acad. Sci. (Math. Sci.), Vol. 99, No. 2, August 1989, pp. 181–190.
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ptotic analysis of some nonlinear problems using Cole transform and spectral theory

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Abstract. We consider initial boundary value problems for certain nonlinear scalar parabolic equations. A formula for the unique classical solution by Hopf-Cole transformations is obtained and the asymptotic behaviour of the solution as time goes to ∞ is studied.

Keywords. Nonlinear parabolic equation; initial boundary value problem; eigenvalue problem; asymptotic behaviour.

oduction

nsider the following initial boundary value problem

$$u_t + \tfrac{1}{2}u_x^2 = \tfrac{1}{2}u_{xx} + q(x) + \mu, \tag{1.1}$$

$$u(x,0) = u_0(x), \tag{1.2}$$

$$u_x(0,t) = a, \tag{1.3}$$

$$u_x(1,t) = b, \tag{1.4}$$

rip $D = \{(x,t): 0 \leqslant x \leqslant 1, t \geqslant 0\}$. μ is a real parameter and a and b are real nts. We assume $q(x)$ is continuous in $0 \leqslant x \leqslant 1$, $u_0(x)$ is twice continuously ntiable in $0 \leqslant x \leqslant 1$, and $(u_0)_x(0) = a$, $(u_0)_x(1) = b$.

ote by $D^0 = \{(x,t): 0 \leqslant x \leqslant 1, t > 0\}$. By a classical solution of (1.1)–(1.4) we function $u(x,t)$ with $u(x,t)$ and $u_x(x,t)$ continuous in D and u_t and u_{xx} continuous which satisfies the partial differential equation (1.1) in D^0 and the initial and ary conditions (1.2)–(1.4) in the usual sense.

g the Hopf-Cole transformation, (see Hopf [2]) we linearize the problem .4) and obtain an expression for the solution in terms of the eigenvalues and nctions of the eigenvalue problem

$$\tfrac{1}{2}\phi_{xx} = (q(x) + \lambda)\phi, \tag{1.5}$$

$$\phi_x(0) + a\phi(0) = 0, \tag{1.6}$$

$$\phi_x(1) + b\phi(1) = 0. \tag{1.7}$$

ove the validity of the expression for $u(x,t)$ and study its asymptotic behaviour

as $t \to \infty$ using the following facts concerning the eigenvalues and eigenfunctions of (1.5)–(1.7), (see Birkhoff and Rota [1]).
(a) The spectrum of (1.5)–(1.7) is discrete and can be ordered

$$\lambda_0 > \lambda_1 > \cdots, \tag{1.8}$$

$$\lambda_n = -\tfrac{1}{2}(n^2\pi^2) + O(1) \quad \text{as } n \to \infty. \tag{1.9}$$

(b) Let $\phi_n(x)$ be the normalized eigenfunctions corresponding to λ_n, the set $\{\phi_n(x), n = 0, 1, 2, \ldots\}$ is a complete set for $L^2[0, 1]$.
Also $\phi_n(x)$ has the following estimates uniformly in $x \in [0, 1]$

$$\phi_n(x) = \sqrt{2}\cos\frac{n\pi}{\sqrt{2}}x + \frac{O(1)}{n} \quad \text{as } n \to \infty, \tag{1.10}$$

$$\phi_x'(x) = -n\pi\sin\frac{n\pi}{\sqrt{2}}x + O(1) \quad \text{as } n \to \infty. \tag{1.11}$$

(c) $\phi_0(x) \neq 0$ $\forall x \in (0, 1)$.
Since $\phi_0(x) \neq 0$ $\forall x \in (0, 1)$ we can assume $\phi_0(x) > 0$ for $x \in (0, 1)$. We claim that

$$\phi_0(x) > 0 \quad \forall x \in [0, 1]. \tag{1.12}$$

If $\phi_0(0) = 0$, then by the boundary condition (1.6) $(\phi_0)_x(0) = 0$, then $\phi_0(x) \equiv 0$, because $\phi_0(x)$ solves

$$\tfrac{1}{2}\phi_{xx} = (q(x) + \lambda)\phi,$$

$$\phi(0) = 0,$$

$$\phi_x(0) = 0,$$

and this has only one solution $\phi(x) \equiv 0$. By the same argument $\phi_0(1)$ is also not equal to 0. The claim is proved. In our discussion we always take $\phi_0(x)$ normalized so that

$$\int_0^1 \phi_0^2(x)\,dx = 1$$

and

$$\phi_0(x) > 0 \quad \forall x \in [0, 1].$$

This paper is organized as follows. In §2 we prove the uniqueness of classical solution of (1.1)–(1.4) and obtain a valid expression for it. The asymptotic behaviour of the solution as $t \to \infty$ is analysed in §3. In §4 are given some comments on the stationary problem and the Dirichlet problem for the Burger's equation is studied in §5.

2. An expression for classical solution of (1.1)–(1.4)

First we prove the uniqueness of classical solution of (1.1)–(1.4) by standard energy estimates. Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions and let $Z(x, t) = u_1(x, t) - u_2(x, t)$.

Then from (1.1)–(1.4) we get, $Z(x,t)$ which is solved as

$$Z_t + \tfrac{1}{2}[(u_1)_x + (u_2)_x]Z_x = \tfrac{1}{2}Z_{xx}, \tag{2.1}$$

$$Z_x(0,t) = 0, \tag{2.2}$$

$$Z_x(1,t) = 0, \tag{2.3}$$

$$Z(x,0) = 0. \tag{2.4}$$

Multiplying (2.1) by Z and integrating by parts w.r.t x, we get using (2.2) and (2.3)

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 Z^2(x,t)\,\mathrm{d}x + \frac{1}{2}\int_0^1 (u_1+u_2)_x Z_x Z\,\mathrm{d}x = -\frac{1}{2}\int_0^1 Z_x^2(x,t)\,\mathrm{d}x. \tag{2.5}$$

Let

$$C_T = \sup_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant t\leqslant T}} |(u_1+u_2)_x|.$$

From (2.5) we obtain, for $0 \leqslant t \leqslant T$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 Z^2(x,t)\,\mathrm{d}x \\
&\leqslant -\int_0^1 Z_x^2(x,t)\,\mathrm{d}x + 2C_T\left(\int_0^1 Z_x^2(x,t)\,\mathrm{d}x\right)^{1/2}\left(\int_0^1 Z^2(x,t)\,\mathrm{d}x\right)^{1/2} \\
&\leqslant -\int_0^1 Z_x^2(x,t)\,\mathrm{d}x + \int_0^1 Z_x^2(x,t)\,\mathrm{d}x + C_T^2\int_0^1 Z^2(x,t)\,\mathrm{d}x \\
&= C_T^2\int_0^1 Z^2(x,t)\,\mathrm{d}x.
\end{aligned}$$

By Grownwall's lemma and (2.4) we get

$$\int_0^1 Z^2(x,t) \leqslant 0 \quad \forall < 0 \leqslant t \leqslant T.$$

Thus $Z^2(x,t)\equiv 0$ for $0\leqslant t\leqslant T$; since T is arbitrary we get $Z(x,t)\equiv 0$ i.e. $u_1(x,t)\equiv u_2(x,t)\forall(x,t)\in D$.

Next we construct the unique classical solution of (1.1)–(1.4). First we need the following:

Lemma 2.1. *Let $v(x,t)$ be the solution of*

$$v_t = \tfrac{1}{2}v_{xx} - (q(x)+\mu)v, \tag{2.6}$$

$$v_x(0,t) + av(0,t) = 0, \tag{2.7}$$

$$v_x(1,t) + bv(1,t) = 0, \tag{2.8}$$

$$v(x,0) = \exp[-u_0(x)]. \tag{2.9}$$

Then

(i) $v(x,t)>0$.

(ii) *The function*

$u(x,t)=-\log v(x,t)$

is a solution of (1.1)–(1.4)

Proof. To prove (i), set

$$v(x,t)=\exp(-Cx+Mt)W(x,t), \tag{2.10}$$

where

$$C=\min(a,b)$$

$$M=1+\sup_{0\leqslant x\leqslant 1}\left|\frac{C^2}{2}-q(x)-\mu\right|.$$

To prove $v(x,t)>0$, it is enough to prove $W(x,t)>0$. We prove this for the case $C=a$, the other case being a similar one.

Assume the contrary, i.e. $\min W(x,t)\leqslant 0$. Now from (2.6)–(2.10) we have $W(x,t)$ which is solved as

$$\tfrac{1}{2}W_{xx}-CW_x+\left(\frac{C^2}{2}-q(x)-\mu-M\right)W-W_t=0, \tag{2.11}$$

$$W_x(0,t)+(a-C)W(0,t)=0, \tag{2.12}$$

$$W_x(1,t)+(b-C)W(1,t)=0, \tag{2.13}$$

$$W(x,0)=\exp[Cx-u_0(x)]. \tag{2.14}$$

Notice that the coefficient of W, $(C^2/2)-q(x)-\mu-M<0$ by the definition of M so that the maximum principle can be applied (see Smoller [3]). The maximum principle min of $W(x,t)$ has to occur at the boundaries and if min is achieved at $x=0$, then $W_x(0,t)>0$ and if it occurs at $x=1$, $W_x(1,t)<0$.

By assumption $\min W(x,t)\leqslant 0$. So evidently min is not achieved for $t=0$. Since $a=C$, $W_x(0,t)=0$ (by (2.12)) so again min cannot be achieved at $x=0$. From (2.13)

$$W_x(1,t)=(a-b)W(1,t).$$

But $a-b\leqslant 0$, which implies that $W(x,t)$ cannot achieve a non-positive minimum at $x=1$. So we have $\min W(x,t)>0$.

To prove (ii) notice that from (i)

$$u(x,t)=-\log v(x,t) \tag{2.15}$$

is well defined. A simple calculation gives

$$u_t=\frac{-\frac{1}{2}v_{xx}+(q(x)+\mu)v}{v},$$

where we used (2.6)

$$u_x = -v_x/v$$
$$u_{xx} = (-vv_{xx} + v_x^2)/v^2. \tag{2.16}$$

It follows that $u(x,t)$ given by (2.15) is solved as

$$u_t + \tfrac{1}{2}u_x^2 = \tfrac{1}{2}u_{xx} + q(x) + \mu.$$

From (2.16), (2.7) and (2.8) we get

$$u_x(0,t) = a,$$
$$u_x(1,t) = b.$$

The proof of lemma is complete.

Let λ_n and $\phi_n(x)$ be as in §1. In the next lemma we get an expression for the solution of (1.1)–(1.4).

Lemma 2.2. *Let* $u(x,t)$ *be the classical solution of* (1.1)–(1.4) *then*

$$u(x,t) = -\log\left[\sum_0^\infty \exp[(\lambda_n - \mu)t]a_n\phi_n(x)\right] \tag{2.17}$$

where

$$a_n = \int_0^1 \exp[-u_0(x)]\phi_n(x)\,dx. \tag{2.18}$$

Proof. It is easy to check that $v(x,t)$ is a solution of (2.6)–(2.9) iff

$$v(x,t) = \exp(-\mu t)V^*(x,t), \tag{2.19}$$

where $V^*(x,t)$ is the solution of

$$V_t^* = \tfrac{1}{2}v_{xx}^* - q(x)V^*,$$
$$V_x^*(0,t) + aV^*(0,t) = 0,$$
$$V_x^*(1,t) + bV^*(1,t) = 0,$$
$$V^*(x,0) = \exp[-u_0(x)] = V_0^*(x).$$

By separation of variable we obtain

$$V^*(x,t) = \sum_0^\infty \exp(\lambda_n t)a_n\phi_n(x) \tag{2.20}$$

where

$$a_n = \int_0^1 \exp[-u_0(x)]\phi_n(x)\,dx.$$

From (1.5), (1.9), (1.10) and (1.11) it follows that for $n > N$, N sufficiently large

$$|\phi_n^{(k)}(x)| \leqslant Cn^k, \quad k = 0, 1, 2$$

$$|\exp(\lambda_n t| \leqslant 1 \quad \forall t \geqslant 0 \tag{2.21}$$

$$|\exp(\lambda_n t)| \leqslant C \exp[-\tfrac{1}{2}(n^2\pi^2)\delta] \quad \forall t \geqslant \delta > 0.$$

$C > 0$ is a constant independent of x, t and n and $\phi_n^{(k)}$ denote the kth derivative w.r.t. x of $\phi_n(x)$. To get an estimate for a_n, notice that $V_0^*(x) = \exp[-u_0(x)]$ satisfies

$$\begin{aligned} &V_x^*(0) + aV^*(0) = 0 \\ &V_x^*(1) + bV^*(1) = 0. \end{aligned} \tag{2.22}$$

Multiplying (1.5) by $V_0^*(x)$, integrating w.r.t. x, integrating by parts and using (2.22), we get

$$\frac{1}{2}\int_0^1 \phi_n(x)(V_0^*)''(x)\,\mathrm{d}x = \lambda_n \int_0^1 \phi_n(x) V_0^*(x)\,\mathrm{d}x.$$

for $n \geqslant N$

$$a_n = \int_0^1 \phi_n(x) V_0^*(x)\,\mathrm{d}x = \frac{1}{2\lambda_n}\int_0^1 (V_0^*)''(x)\phi_n(x)\,\mathrm{d}x$$

so that, for $n > N$

$$|a_n| \leqslant \frac{C}{n^2}|b_n|, \tag{2.23}$$

where

$$b_n = \int_0^1 (V_0^*)''(x)\phi_n(x)\,\mathrm{d}x.$$

From (2.21) and (2.23) it follows that $V^*(x, t)$ is continuous in D. Also

$$\begin{aligned} V_x^*(x,t) &= \sum_0^\infty \exp(\lambda_n t) a_n \phi_n'(x) \\ &= \sum_0^{N-1} \exp(\lambda_n t) a_n \phi_n'(x) + \sum_N^\infty \exp(\lambda_n t) a_n \phi_n'(x). \end{aligned} \tag{2.24}$$

Now

$$\left|\sum_N^\infty \exp(\lambda_n t) a_n \phi_n'(x)\right| \leqslant C^2 \sum_N^\infty \frac{1}{n^2}|b_n|\cdot n \tag{2.25}$$

and

$$\sum_0^\infty |b_n|^2 = \int_0^1 (V_0^*)''(x)^2\,\mathrm{d}x \tag{2.26}$$

by completeness of eigenfunctions. From (2.24)–(2.26) it follows that

$$|V_x^*(x,t)| \leqslant \sum_0^{N-1} \exp(\lambda_n t) a_n \phi_n'(x) + C^2\left(\sum_N^\infty \frac{1}{n^2}\right)^{1/2}\left(\sum_N^\infty |b_n|^2\right)^{1/2}$$

so that the series is absolutely convergent and hence $V_x^*(x, t)$ is continuous in D.

Using the estimates (2.21), (2.23) and (2.26) and using the same argument as before, $v_{xx}^*(x, t)$ and $V_t^*(x, t)$ are also continuous in $0 \leqslant x \leqslant 1$, $t \geqslant \delta > 0$. Now by (2.19)

$$v(x,t) = \exp(-\mu t) V^*(x,t) = \sum_0^\infty \exp[(\lambda_n - \mu)t] a_n \phi_n(x)$$

and by lemma 2.1

$$u(x,t) = -\log\left(\sum_0^\infty \exp[(\lambda_n - \mu)t]a_n\phi_n(x)\right)$$

is the classical solution of (1.1)–(1.4). The proof of lemma (2.2) is complete.

Next we study the asymptotic behaviour of the solution $u(x,t)$ constructed in this section.

3. Asymptotic behaviour of the classical solution of (1.1)–(1.4)

In this section $u(x,t)$ denotes the unique classical solution of (1.1)–(1.4) constructed in §2. We shall prove the following.

Theorem. (i) *Let $\mu = \lambda_0$; then*

$$\sup_{0 \leqslant x \leqslant 1} |u(x,t) + \log(a_0\phi_0(x))| \leqslant C\exp[(\lambda_1 - \lambda_0)t].$$

(ii) *Let $\mu \neq \lambda_0$; then for $t \geqslant 1$*

$$\sup_{0 \leqslant x \leqslant 1} \left|\frac{u(x,t)}{t} + (\lambda_0 - \mu) + \frac{\log(a_0\phi_0(x))}{t}\right| \leqslant \frac{C}{t}\exp[(\lambda_1 - \lambda_0)t].$$

where λ_0 and $\phi_0(x)$ are as in §1, a_0 is given by (2.18) and C a positive constant independent of x and t.

Proof. From lemma 2.2 we have

$$u(x,t) = -\log\left\{\sum_0^\infty \exp[(\lambda_n - \mu)t]a_n\phi_n(x)\right\}.$$

To prove (i), notice that when $\mu = \lambda_0$, the above expression for $u(x,t)$ becomes

$$u(x,t) = -\log\left[a_0\phi_0(x) + \sum_1^\infty \exp[(\lambda_n - \lambda_0)t]a_n\phi_n(x)\right]$$

$$= -\log[a_0\phi_0(x)] - \log\left[1 + \sum_1^\infty \exp[(\lambda_n - \lambda_0)t]\left(\frac{a_n}{a_0}\right)\left(\frac{\phi_n(x)}{\phi_0(x)}\right)\right]$$

$a_0 > 0$ and $\inf_{0 \leqslant x \leqslant 1} \phi_0(x) > 0$ by (1.12). Using the estimate (2.21) we have, for $t \geqslant 1$

$$\left|\sum_1^\infty \exp[(\lambda_n - \lambda_0)t]\left(\frac{a_n}{a_0}\right)\frac{\phi_n(x)}{\phi_0(x)}\right| \leqslant C\exp[(\lambda_1 - \lambda_0)t]$$

so that we get

$$|u(x,t) + \log[a_0\phi_0(x)]| \leqslant \log\{1 + C\exp[(\lambda_1 - \lambda_0)t]\}.$$

But for $0 < y < 1$

$$\log(1 + y) \leqslant y$$

we get

$$|u(x,t) + \log(a_0\phi_0(x)| \leqslant C\exp[(\lambda_1 - \lambda_0)t].$$

To prove (ii) arguing as before, we get

$$u(x,t) = -\log\{\exp[(\lambda_0 - \mu)t]a_0\phi_0(x)\} - \log\left[1 + \sum_1^\infty \exp[(\lambda_n - \lambda_0)t]\left(\frac{a_n}{a_0}\right)\left(\frac{\phi_n(x)}{\phi_0(x)}\right)\right] \tag{3.4}$$

As before, for $t \geqslant 1$,

$$\left|\log\left[1 + \sum_1^\infty \exp[(\lambda_n - \lambda_0)t]\left(\frac{a_n}{a_0}\right)\frac{\phi_n(x)}{\phi_0(x)}\right]\right| \leqslant C\exp[(\lambda_1 - \lambda_0)t]. \tag{3.5}$$

Also

$$\log\{\exp[(\lambda_0 - \mu)t]a_0\phi_0(x)\} = (\lambda_0 - \mu)t + \log(a_0\phi_0(x)). \tag{3.6}$$

Using (3.5) and (3.6) in (3.4) we get for $t \geqslant 1$

$$\sup_{0\leqslant x\leqslant 1}\left|\frac{u(x,t)}{t} + (\lambda_0 - \mu) + \frac{\log(a_0\phi_0(x))}{t}\right| \leqslant \frac{C}{t}\exp[(\lambda_1 - \lambda_0)t].$$

The proof of the theorem is complete.

4. Some remarks on the stationary problem

Consider the stationary problem

$$p_x^2/2 = \tfrac{1}{2}p_{xx} + q(x) + \lambda_0, \tag{4.1}$$

$$p_x(0) = a, \tag{4.2}$$

$$p_x(1) = b, \tag{4.3}$$

where λ_0 is as in §1. Consider the one-parameter family of functions

$$p_\alpha(x) = -\log(\alpha\phi_0(x)), \ \alpha > 0 \tag{4.4}$$

$\phi_0(x)$ is as in §1 with condition (1.12). By a direct calculation it is easy to verify that $p_\alpha(x)$ is a solution of (4.1)–(4.3) for each $\alpha > 0$. The theorem in §3, part (i) says that, in the case $\mu = \lambda_0$, the solution $u(x,t)$ of (4.1)–(4.4) converges to $p_{a_0}(x)$.

The second part of the theorem says that if $\mu \neq \lambda_0$, $u(x,t) \to \infty$ as $t \to \infty$. The stationary problem

$$p_x^2/2 = \tfrac{1}{2}p_{xx} + q(x) + \mu, \tag{4.5}$$

$$p_x(0) = a, \tag{4.6}$$

$$p_x(1) = b \tag{4.7}$$

does not have any solution if $\mu \neq \lambda_0$. In fact as in lemma 2.1 one can easily check

that $p(x)$ is a solution to (4.5)–(4.7) iff the function

$$h = \exp(-p) \tag{4.8}$$

is a solution to

$$\tfrac{1}{2}h_{xx} = [q(x) + \mu]h, \tag{4.9}$$

$$h_x(0) + ah(0) = 0, \tag{4.10}$$

$$h_x(1) + bh(1) = 0. \tag{4.11}$$

But (4.9)–(4.11) has a non-zero solution iff $\mu = \lambda_n$, $n = 0, 1, 2, \ldots$, and λ_n is as in §1. Further the corresponding solution has to be positive, by (4.8). This happen iff $\mu = \lambda_0$.

5. Burger's equation

The method presented in previous sections can be used to get a closed form expression and asymptotic behaviour of the solution of the Burger's equation in $D = \{|x, t): a \leqslant x \leqslant 1, t \geqslant 0\}$. Let $u(x, t)$ be the unique solution to

$$u_t + (u^2/2)_x = \tfrac{1}{2}u_{xx}, \tag{5.1}$$

$$u(x, 0) = u_0(x), \tag{5.2}$$

$$u(0, t) = a, \tag{5.3}$$

$$u(1, t) = b, \tag{5.4}$$

a and b are constants, $u_0(x)$ is $C^1[0, 1]$ and $u_0(0) = a$, $u_0(1) = b$. Then it can be easily seen as in lemmas 2.1 and 2.2 that

$$u(x, t) = -\frac{\sum_0^\infty \exp(\lambda_n t) a_n \phi_n^1(x)}{\sum_0^\infty \exp(\lambda_n t) a_n \phi_n(x)} \tag{5.5}$$

$\lambda_0 > \lambda_1 > \cdots \to -\infty$ are the eigenvalues of

$$\tfrac{1}{2}\phi_{xx} = \lambda\phi,$$

$$\phi_x(0) + a\phi(0) = 0,$$

$$\phi_x(1) + b\phi(1) = 0$$

and $\phi_n(x)$, $n = 0, 1, \ldots$ are the normalized eigenfunctions corresponding to λ_n with $\phi_0(x) > 0$ and

$$a_n = \int_0^1 \exp\left(-\int_0^x u_0(y)\,\mathrm{d}y\right)\phi_n(x)\,\mathrm{d}x. \tag{5.6}$$

As in §3, it is easy to prove the following.

Theorem. *Let $u(x,t)$ be the solution of* (5.1)–(5.4); *then*

$$u(x,t) = -[\log \phi_0(x)]_x + O[\exp[(\lambda_1 - \lambda_0)t]$$

uniformly in $x \in [0,1]$.

Proof. From (5.5) we have,

$$u(x,t) = -\frac{\left[\phi_0^1(x) + \sum_1^\infty \exp[(\lambda_n - \lambda_0)t]\left(\frac{a_n}{a_0}\right)\phi_n^1(x)\right]}{\phi_0(x)}$$

$$\times \left[1 + \sum_1^\infty \exp[(\lambda_n - \lambda_0)^t]\left(\frac{a_n}{a_0}\right)\left(\frac{\phi_n}{\phi_0}\right)\right]^{-1}$$

$$= -\frac{\left[\phi_0^1(x) + \sum_1^\infty (\exp[(\lambda_n - \lambda_0)t])\right]}{\phi_0(x)}$$

$$\times [1 + O(\exp[(\lambda_1 - \lambda_0)t])]^{-1} \text{ uniformly in } x$$

$$= -\frac{\phi_0^1(x)}{\phi_0(x)}[1 + O(\exp[(\lambda_1 - \lambda_0)t])]$$

uniformly in $x \in [0,1]$. The proof of theorem is complete. q.e.d.

References

[1] Birkhoff Garrett and Rota Gian-carlo, *ordinary differential equations,* (John Wiley and Sons) 3rd edition (1978)

[2] Hopf E, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Commun. Pure Appl. Math.* **13** (1950) 201–230

[3] Smoller J, *Shock waves and reaction diffusion equations* (New York: Springer-Verlag) (1983)

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Cycles on the generic abelian threefold

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Abstract. H Clemens and C Schoen gave examples of three-folds where the group of codimension two cycles modulo algebraic equivalence has infinite rank. This paper provides yet another example of the same phenomenon.

Keywords. Algebraic cycles; generic curves; generic Abelian varieties; symplectic group action.

Let X be a smooth projective variety over C and denote by $R^2(X)$ the group of codimension two algebraic cycles homologically equivalent to zero modulo the subgroup of those cycles algebraically equivalent to zero. The first example of a variety X for which $R^2(X)\otimes Q\neq 0$ was given by Griffiths (see [5]). More recently, Clemens showed that $R^2(X)\otimes Q$ is infinite-dimensional where X is the generic quintic hypersurface in P^4 (see [3]).

We shall show in this paper that $R^2(X)\otimes Q$ is infinite dimensional where X is the generic abelian variety of dimension three. This result follows easily from a group-theoretic argument and the following basic result of Ceresa, which we now explain.

Denote by $\phi\colon C\to J(C)$ the embedding of a curve C (well-defined up to translations) in its Jacobian. Put $\psi(x)=-\phi(x)$, for $x\in C$ and consider the cycle $S(C)=[\phi(C)]-[\psi(C)]$ which is obviously trivial homologically. The theorem of Ceresa (see [2]) is that $S(C)$ is non-zero in the group $R^2(J(C))\otimes Q$ where C is the generic curve of genus three.

To construct cycles on a fixed three-dimensional abelian variety A we may proceed as follows. First choose an isogeny $h\colon B\to A$ with B principally polarized. Then $B\cong J(C)$ for some curve C of genus three (possibly degenerate) and thus B has the Ceresa cycle $S(C)$. Its image $h_*S(C)\in R^2(A)\otimes Q$ is non-zero if A is generic, thanks to the result of Ceresa. Keeping A fixed, there are plenty of choices of h and each of these choices gives a cycle on A. These cycles are linearly independent because they twist by different characters of $\mathrm{Gal}(\bar{F}/F)$ where F is a field of definition of A.

The formal argument follows.

Good references for all the facts concerning the moduli spaces that we have used in this paper are [4], [6] and [7].

Let N be a natural number $\geqslant 3$.

Denote by $M(N)$ (resp. $X(N)$) the moduli of curves (resp. principally polarized

abelian varieties) of genus (resp. dimension) three with level N structure, defined over C, the field of complex numbers. These are smooth irreducible varieties of dimension six, and let $\mathscr{C}(N)\to M(N)$ and $\mathscr{A}(N)\to X(N)$ be the universal family of curves and abelian varieties respectively. We then have the $\mathrm{Sp}(6;Z/N)$-equivariant commutative diagram:

$$\text{I}\qquad \begin{array}{ccc} J(\mathscr{C}(N)) & \longrightarrow & \mathscr{A}(N) \\ \downarrow & & \downarrow \\ M(N) & \longrightarrow & X(N) \end{array}$$

where $J(\mathscr{C}(N))$ is the family of Jacobians of $\mathscr{C}(N)\to M(N)$. Note that $-1\in\mathrm{Sp}(6;Z/N)$ acts trivially on $X(N)$ and its action on $\mathscr{A}(N)$ is simply $x\to -x$ on the fibres of $\mathscr{A}(N)\to X(N)$.

Now $\mathrm{Sp}(6;Z/N)/\{\pm 1\}$ acts faithfully on $X(N)$, while the action of $\{\pm 1\}$ on $M(N)$ is non-trivial. Applying the global Torelli theorem, we see that

$$\{\pm 1\}\backslash M(N)\to X(N)$$

is a birational isomorphism (in fact it is an open immersion by Zariski's Main Theorem).

Let $C(N)$ and $A(N)$ be the generic fibres of $\mathscr{C}(N)\to M(N)$ and $\mathscr{A}(N)\to X(N)$. These are varieties over the functon fields $E(N)$ of $M(N)$ and $F(N)$ of $X(N)$ respectively. Furthermore $[E(N):F(N)]=2$. From I we get a $\mathrm{Sp}(6;\ Z/N)$-equivariant commutative diagram:

$$\text{II}\qquad \begin{array}{ccc} J(C(N)) & \longrightarrow & A(N) \\ \downarrow & & \downarrow \\ \operatorname{Spec} E(N) & \longrightarrow & \operatorname{Spec} F(N) \end{array}$$

If N_1 divides N_2, there is a $\mathrm{Sp}(6;N_2)$-equivariant commutative diagram:

$$\begin{array}{ccc} \mathscr{C}(N_2) & \longrightarrow & \mathscr{C}(N_1) \\ \downarrow & & \downarrow \\ M(N_2) & \longrightarrow & M(N_1) \end{array}$$

so that if E is the union of all the $E(N)$, we get a curve $C\to\operatorname{Spec} E$ with the action of $\mathrm{Sp}(6;\hat{Z})$. Similarly we get an abelian variety $A\to\operatorname{Spec} F$, where F is the union of all the $F(N)$. Finally II now gives a $\mathrm{Spec}(6;\hat{Z})$-equivariant commutative diagram:

$$\text{III}\qquad \begin{array}{ccc} J(C) & \longrightarrow & A \\ \downarrow & & \downarrow \\ \operatorname{Spec} E & \longrightarrow & \operatorname{Spec} F \end{array}$$

Note also that $[E:F]=2$, and that III gives an isomorphism:

$$J(C)\to A_E.$$

Factoring $C\to\operatorname{Spec} E$ by the action of $-1\in\mathrm{Sp}(6;\hat{Z})$, we get a curve $C'\to\operatorname{Spec} F$ such that $C'_E\cong C$. Put $A'=J(C')$. Thus we have (non-isomorphic) abelian varieties A'

and A defined over F and isomorphisms

$$A'_E \leftarrow J(C) \rightarrow A_E.$$

Denote by $f\colon A'_E \rightarrow A_E$ the induced isomorphism and by σ the non-trivial element of $\mathrm{Gal}(E/F)$. The equivalence of III under $-1 \in \mathrm{Sp}(6;\hat{Z})$ is now equivalent to

IV $\qquad f\circ(1_{A'} \times \sigma) = (i_A \times \sigma)\circ f.$

In IV above and always, $i_Z\colon Z \rightarrow Z$ denotes the morphism $x \mapsto -x$ of an abelian variety Z. Occasionally i_Z will be abbreviated to i simply.

We are now ready to tackle the Galois action on the generic Ceresa cycle. We need first some notation and an elementary result.

The group of codimension k cycles modulo algebraic equivalence on a variety X will be denoted by $B^k(X)$. If X is defined over K and L is a field extension of K, then $\mathrm{Aut}(L/K)$ acts on $B^k(X_L)$.

Lemma. Let D be a curve of genus $g \geqslant 1$ over a field K. Denote by $\bar{K}$ an algebraic closure of K. The Ceresa cycle $S(D) \in B^{g-1}(J(D)_{\bar{K}})$ is invariant under the action of $\mathrm{Gal}(\bar{K}/K)$, *and $i^*S(D) = -S(D)$.*

Proof. A divisor R of degree $= -1$ on $D_{\bar{K}}$ defines an embedding

$$\phi_R\colon D_{\bar{K}} \rightarrow J(D)_{\bar{K}}$$

given by $\phi_R(x) = [x] + R$. If S is also a divisor of degree $= -1$, then ϕ_S is a translate of ϕ_R and therefore the cycle $\xi = \phi_R(D_{\bar{K}})$ as an element of $B^{g-1}(J(D)_{\bar{K}})$ does not depend on the choice of R at all. For all $g \in \mathrm{Gal}(\bar{K}/K)$, $g(\phi_R) = \phi_{gR}$ and this shows that ξ is invariant under the Galois action. Since the Ceresa cycle $S(D)$ is $\xi - i^*(\xi)$, it follows that $S(D)$ is invariant too. Also

$$i^*S(D) = i^*(\xi) - \xi = -S(D)$$

and this proves the lemma.

Now let $\bar{F} \supset E \supset F$ be an algebraic closure of F and let χ be the composite $\mathrm{Gal}(\bar{F}/F) \rightarrow \mathrm{Gal}(E/F) \cong \{\pm 1\}$. Let $f_F\colon A'_F \rightarrow A_F$ be extended from $f\colon A'_E \rightarrow A_E$ and put $\theta = f_F(S(C')) \in B^2(A_F)$.

PROPOSITION 1

For all $g \in \mathrm{Gal}(\bar{F}/F)$, $g^\theta = \chi(g)\theta$.

Proof. Abbreviating i_A, $i_{A'}$, etc. to i, and because $i\circ f_{\bar{F}} = f_{\bar{F}}\circ i$, we deduce from IV that for all $g \in \mathrm{Gal}(\bar{F}/F)$,

(a) $f_F\circ(1_{A'} \times g) = (1_A \times g)\circ f_F$ if $\chi(g) = 1$, and

(b) $f_F\circ(i_{A'} \times g) = (1_A \times g)\circ f_F$ if $\chi(g) = -1$. For an element Z of $B^k(A'_F)$ this gives

(c) $f_F(gZ) = gf_F(Z)$ if $\chi(g) = 1$

(d) $f_F(g(i^*Z)) = gf'_F(Z)$ if $\chi(g) = -1$.

Putting $Z = S(C')$ and applying the above lemma, the proposition follows.

We shall now embed $\mathrm{Gal}(\bar{F}/F)$ in a larger group that acts on $B^k(A_{\bar{F}})\otimes Q$. Recall that $X(N)$ is the quotient of the Siegel half-space $H = \{T \in M_3(C)\colon T = {}^tT \text{ and } \mathrm{Im}\, T > 0\}$ by the action of $\Gamma(N)$, the principal congruence subgroup of level N in $\mathrm{Sp}(6; Z)$. Let $\widetilde{\mathrm{Sp}}(6; R)$ be the subgroup of $\mathrm{GL}_6(R)$ generated by $\mathrm{Sp}(6; R)$ and the scalar matrices. Put $\widetilde{\mathrm{Sp}}(6; Q) = \widetilde{\mathrm{Sp}}(6; R) \cap \mathrm{GL}_6(Q)$. There is an action of $\widetilde{\mathrm{Sp}}(6; R)/R^*$ on H. For every $g \in \widetilde{\mathrm{Sp}}(6; Q)$ there is a natural number $\mathbf{a}$ and a commutative diagram:

$$\begin{array}{ccc} H & \xrightarrow{g} & H \\ \downarrow & & \downarrow \\ X(N\mathbf{a}) & \longrightarrow & X(N) \end{array}$$

where the vertical arrows are the quotient maps. Passing to the direct limit over the N and taking generic points we get, for each $g \in \widetilde{\mathrm{Sp}}(6; Q)$ an automorphism $\rho_1(g)$ of $\mathrm{Spec}\, F$. And $\rho_1(g) = 1_F$ if $g \in Q^*$.

Moreover, if $g \in M_6(Z) \cap \widetilde{\mathrm{Sp}}(6; Q)$ we get commutative diagrams:

$$\begin{array}{ccc} \mathscr{A}(N\mathbf{a}) & \longrightarrow & \mathscr{A}(N) \\ \downarrow & & \downarrow \\ X(N\mathbf{a}) & \longrightarrow & X(N) \end{array}$$

where the horizontal arrows are isogenies on the fibres. These induce:

V
$$\begin{array}{ccc} A_F & \xrightarrow{\rho_2(g)} & A_F \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, F & \xrightarrow{\rho_1(g)} & \mathrm{Spec}\, F \end{array}$$

Denote by $j\colon \mathrm{Spec}\, \bar{F} \to \mathrm{Spec}\, F$ the given morphism and define

$$G = \{(\alpha, g) \in \mathrm{Aut}\,\mathrm{Spec}\, \bar{F} \times \widetilde{\mathrm{Sp}}(6; Q) \mid \rho_1(g) \circ j = j \circ \alpha\},$$

$$S = \{(\alpha, g) \in G \mid g \in M_6(Z)\}$$

$$T = \{(\alpha, g) \in S \mid g \text{ is a scalar matrix and } \alpha = 1_{\bar{F}}\}.$$

Then we have an exact sequence:

(A) $1 \to \mathrm{Gal}(\bar{F}/F) \to G \to \widetilde{\mathrm{Sp}}(6; Q) \to 1.$

Also

(B) $G = S^{-1} \cdot T$ and S^{-1} is a semi-group, and

(C) T is contained in the centre of G.

From (B) and (C) it follows that any homomorphism of S^{-1} to a group extends uniquely to a homomorphism of G. To define $\rho\colon G \to \mathrm{Aut}\, B^k(A_{\bar{F}}) \otimes Q$ it suffices therefore to give ρ on S^{-1} such that $\rho(\omega_1)\rho(\omega_2) = \rho(\omega_1 \omega_2)$ for all $\omega_1, \omega_2 \in S^{-1}$.

Now let $(\alpha, g) \in S$. Taking the fibre-product of

$$\begin{array}{ccc} \operatorname{Spec}\bar{F} & \xrightarrow{\alpha} & \operatorname{Spec}\bar{F} \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ \operatorname{Spec}F & \xrightarrow{\rho_1(g)} & \operatorname{Spec}F \end{array}$$

with V, we get

$$\begin{array}{ccc} A_{\bar{F}} & \xrightarrow{s} & A_{\bar{F}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\bar{F} & \xrightarrow{\alpha} & \operatorname{Spec}\bar{F} \end{array}$$

so that the induced morphism $A_{\bar{F}} \to \alpha^* A_{\bar{F}}$ is an isogeny. From the easy isogeny lemma below it follows that

$$s^*: B^k(A_{\bar{F}}) \otimes Q \to B^k(A_{\bar{F}}) \otimes Q$$

is an isomorphism. We define $\rho(s^{-1}) = s^*$. That ρ is an action on S^{-1} follows from the fact: $\rho_2(g_1 g_2) = \rho_2(g_1)\rho_2(g_2)$ for $g_1, g_2 \in M_6(Z) \cap \widetilde{\mathrm{S}}\mathrm{p}(6; Q)$. Modulo the lemma below, therefore, an action ρ of G on $B^k(A_{\bar{F}}) \otimes Q$ has been defined.

Isogeny lemma. If $f: X \to Y$ is an isogeny of abelian varieties, $f^*: B^k(Y) \otimes Q \to B^k(X) \otimes Q$ is an isomorphism.

Proof. In fact $(1/d)f_*$ is the inverse of f^*, where d is the degree of the isogeny. The projection formula gives $f_* f^* Z = \mathrm{d}Z$. And $f^* f_* W$, being the sum of the translates of W by the elements of the kernel, is algebraically equivalent to $\mathrm{d}W$, and this proves the lemma.

Theorem. *$B^2(A_{\bar{F}}) \otimes Q$ and $R^2(A_{\bar{F}}) \otimes Q$ are infinite-dimensional.*

$R^2(A_{\bar{F}}) \otimes Q$, consisting of homologically trivial cycles, has finite codimension in $B^2(A_{\bar{F}}) \otimes Q$, and so both the assertions of the theorem are equivalent.

Choose a sequence $r_1, r_2, \ldots \in \mathrm{Sp}(6; Q)$ which form a system of coset-representatives for $\mathrm{Sp}(6, Z) \backslash \mathrm{Sp}(6, Q)$ and then lift the r_i to $s_i \in G$. The infinite-dimensionality of $B^2(A_{\bar{F}}) \otimes Q$ follows from the linear independence of $\rho_1(s_1)\theta$, $\rho(s_2)\theta, \ldots,$ with θ as in Proposition 1.

From Proposition 1 it follows that $\rho(h)\rho(g)\theta = \chi^g(h)\rho(g)\theta$ for all $h \in \mathrm{Gal}(\bar{F}/F)$, $g \in G$, where $\chi^g(h) = \chi(g^{-1}hg)$. We shall show that the χ^{s_i} are distinct characters of $\mathrm{Gal}(\bar{F}/F)$, from which the linear independence of the $\rho(s_i)\theta$ follows.

To this end we shall define a closed analytic subset $R(\eta)$ of the Siegel half-space H for any character $\eta: \mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}$ and then show that the $R(\chi^{r_i})$ are all distinct.

A character $\eta = \mathrm{Gal}(\bar{F}/F) \to \{\pm 1\}$ determines a quadratic extension L of F, which gives for some $N \geqslant 3$ a quadratic extension $L(N)$ of $F(N)$ such that $L(N) \cdot F = L$. Now

the $L(N)$ gives a double covering $Y(N) \to X(N)$ and let $R(N) \subset X(N)$ be its branch-locus. Finally let R be the inverse image of $R(N)$ in the projection $H \to X(N)$. That R is independent of the particular choice of N and $L(N)$ is clear. Thus we put $R = R(\eta)$. It is also immediate that $R(\eta^g) = h^{-1}R(\eta)$ where $g \in G$ and h is the image of g in $\widetilde{\mathrm{Sp}}(6; Q)$.

If we take $\eta = \chi$, we have explicitly $L(N) = E(N)$ and $R(\chi)$ is the locus of the hyperelliptic Jacobians (and also the degenerate Jacobians) in the Siegel half-space H. Thus to finish the proof of the theorem, we only need the following.

Lemma. $\pi = \{g \in \mathrm{Sp}(6; R) \doteq gR(\chi) = R(\chi)\}$ *equals* $\mathrm{Sp}(6; Z)$.

Proof. Clearly π is a *closed* subgroup of $\mathrm{Sp}(6; R)$, and its Lie algebra, being stable under the adjoint action of $\pi \supset \mathrm{Sp}(6; Z)$, must be zero or all of Lie $\mathrm{Sp}(6; R)$. In the latter case, $\pi = \mathrm{Sp}(6; R)$ and therefore $R(\chi) = \phi$ or $R(\chi) = H$, which is absurd. In the former, π is discrete and since $\mathrm{Sp}(6; Z)$ is a maximal discrete subgroup of $\mathrm{Sp}(6; R)$, we deduce that $\pi = \mathrm{Sp}(6; Z)$.

This completes the proof of the theorem.

What the above argument gives more generally is the following statement: if $v \in B^2(A_{\bar{F}}) \otimes Q$ is not invariant under $\mathrm{Gal}(\bar{F}/F)$, then the G-orbit of v spans an infinite-dimensional subspace. After all, the $R(\eta)$ can be defined for any irreducible representation η of $\mathrm{Gal}(\bar{F}/F)$, and all that was used is the fact that $R(\eta)$ is non-empty if η is not the trivial representation. In other words, if L is a finite extension of F, the branch locus (as a subset of H) is non-empty—this is a consequence of the congruence subgroup theorem for $\mathrm{Sp}(2n; Z)$, see e.g. [1].

The following interesting question has been raised by Clemens: Is $B^2(A_{\bar{F}}) \otimes Q$ a finitely generated G-module?

References

[1] Bass H, Milnor J and Serre J P, Solution of the Congruence Subgroup Problem *Publ. I.H.E.S.* **33** (1967) 59–137

[2] Ceresa G, C is not algebraically equivalent to C^- in its Jacobian, *Ann. Math.*, **117** (1983) 285–291

[3] Clemens H, Homological equivalence modulo algebraic equivalence is not finitely generated *Publ. I.H.E.S.* **58** (1983) pp. 19–38

[4] Deligne P and Mumford D, The Irreducibility of the space of curves of a given genus *Publ. I.H.E.S.* **36** 75–110

[5] Griffiths P A, On the periods of certain rational integrals, *Ann. Math.*, **90** (1969) 460–541

[6] Igusa J, *Theta Functions*, Springer-Verlag (1972)

[7] Mumford D, *Geometric Invariant Theory*, Springer-Verlag (1965)

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Mumford's example and a general construction

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Abstract. We construct a line bundle on a complex projective manifold (a general ruled variety over a curve) which is not ample, but whose restriction to every proper subvariety is ample. This example is of interest in connection with ampleness questions of vector bundles on varieties of dimension greater than one. The method of construction shows that a stable bundle of positive degree on a curve is ample. The example can be used to show that there is no restriction theorem for Bogomolov stability.

Keywords. Ampleness; stability; vector bundles.

1. Introduction

There is a well-known example of Mumford of a nonample line bundle on a ruled surface which is ample when restricted to any integral curve in the surface (example (10.6), Chapter I in [2]). We present a general construction of a line bundle on a complex projective manifold (of every dimension) which is not ample, but whose restriction to every proper subvariety is ample (Theorem 6.1). To a subvariety of the projective bundle defined by a vector bundle on a curve, we associate an effective divisor in a suitable Grassmannian bundle, by a procedure analogous to the construction of the Chow form. This in turn gives a line subbundle of an associated bundle on the curve and the stability of this bundle enables us to verify Nakai's criterion on the projective bundle. The method of construction also yields a new proof (cf. [3]) that a stable vector bundle of positive degree on a compact Riemann surface is ample. The nonample line bundle constructed can be used to produce an example of a rank two vector bundle which is Bogomolov stable, but whose restriction to proper subvarieties is Bogomolov instable, invalidating a restriction theorem (similar to that of Mehta and Ramanathan for Mumford stability) for Bogomolov stability (see §8).

2. Preliminaries

Let C be a compact Riemann surface. Let $V \to C$ be a holomorphic vector bundle on C of rank r. Let $\mathbb{P}(V)$ be the projective bundle of line quotients of V. If $\pi\colon \mathbb{P}(V) \to C$ is the projection, there is an exact sequence

$$0 \to S \to \pi^* V \to \mathcal{O}_{\mathbb{P}(V)}(1) \to 0$$

on $\mathbb{P}(V)$, where $\mathcal{O}_{\mathbb{P}(V)}(1)$ restricts to $\mathcal{O}(1)$ along the fibres of π.

Let $\mathrm{Gr}(\ell, V)$ be the Grassmannian bundle of ℓ dimensional quotients of V $(0 < \ell < r)$ and $p: \mathrm{Gr}(\ell, V) \to C$ the projection. There is the universal exact sequence of vector bundles

$$0 \to S \to p^*V \to Q \to 0$$

on $\mathrm{Gr}(\ell, V)$, where Q is a vector bundle of rank ℓ on $\mathrm{Gr}(\ell, V)$ which restricts to the universal quotient bundle on the fibres of p. We will need (see Chapter 16, Theorem 2.5, in [4]).

Lemma 2.1. *Let $V \to C$ be a complex vector bundle of rank r. Let $\tau = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$. The homomorphism $\pi^*: H^*(C, \mathbb{Z}) \to H^*(\mathbb{P}(V), \mathbb{Z})$ is an inclusion and $H^*(\mathbb{P}(V), \mathbb{Z})$ is generated as an algebra over $H^*(C, \mathbb{Z})$ by τ, with a single relation*

$$\tau^r - \tau^{r-1}\pi^* c_1(V) = 0,$$

and

$$H^*(\mathbb{P}(V), Z) \cong \bigoplus_{i=0}^{r-1} H^*(C, \mathbb{Z}) \cdot \tau^i.$$

A similar result is true for $\mathrm{Gr}(\ell, V)$ whose cohomology is generated by the Chern classes of the universal quotient bundle Q over the cohomology of C.

3. The ruled variety

Let C be a compact Riemann surface of genus $g \geq 2$. We construct a vector bundle $V \to C$ of rank r (for any $r \geq 2$) such that V is stable, $\det V = \Lambda^r V$ is trivial (such a bundle is necessarily defined by an irreducible representation of $\pi_1(C)$ in $\mathrm{SU}(r)$, by a theorem of Narasimhan-Seshadri), and for any irreducible representation of $\mathrm{SL}(r, \mathbb{C})$ the associated vector bundle remains stable. We require

Lemma 3.1. *Two "general" elements of a connected compact semisimple real Lie group G generate a dense subgroup of G.*

Proof. This argument is essentially due to Madhav Nori. If G is a torus, we can choose a general element to generate a dense subgroup. Otherwise, let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ a Cartan subalgebra. Let

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \text{ a root}} \mathfrak{g}^\alpha$$

be the root space decomposition of $\mathfrak{g}$. Let $\mathfrak{l}$ be a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then we have a decomposition

$$\mathfrak{l} = \mathfrak{h} + \sum \mathfrak{l}^\alpha$$

where each $\mathfrak{l}^\alpha$ is irreducible for $\mathfrak{h}$, and hence has to be one of the $\mathfrak{g}^\alpha$. This shows there are only finitely many such $\mathfrak{l}$. Let

$$A = \{\mathfrak{l} \text{ such that } \mathfrak{l} \text{ is a proper Lie subalgebra of } \mathfrak{g} \text{ with } \mathfrak{h} \subset \mathfrak{l}\}.$$

Then A is a finite set, and $\mathfrak{g} - \bigcup_{\mathfrak{l} \in A} \mathfrak{l}$ is a nonempty open subset of $\mathfrak{g}$. Let X be an element of $\mathfrak{h}$ such that $\exp(X)$ generates a dense subgroup of the maximal torus T corresponding to $\mathfrak{h}$, and Y an element of $\mathfrak{g} - \bigcup_{\mathfrak{l} \in A} \mathfrak{l}$. Let G' be the closure in G of the group generated by $\exp(X)$ and $\exp(Y)$. Then G' contains T. If $\mathfrak{g}'$ is the Lie algebra of G', then $\mathfrak{g}'$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. If $\mathfrak{g}'$ is a proper Lie subalgebra, then $\mathfrak{g}'$ is in A, while Y belongs to $\mathfrak{g}'$, which contradicts the choice of Y. Therefore $\mathfrak{g}' = \mathfrak{g}$. Since G is connected, $G' = G$ and $\exp(X)$ and $\exp(Y)$ generate a dense subgroup. It only remains to observe that the choice of X and Y is general to conclude the proof.

Lemma 3.2. *Let C be a compact Riemann surface of genus $g \geqslant 2$. There exists a representation $\rho: \pi_1(C) \to \mathrm{SU}(r)$, for any $r \geqslant 2$, such that the image of ρ is dense in* $\mathrm{SU}(r)$ *(in fact, a general representation satisfies this).*

Proof. From Lemma 3.1, two general elements of any connected compact semi-simple real Lie group (in particular, $\mathrm{SU}(r)$) generate a dense subgroup. Let α and β be two such elements of $\mathrm{SU}(r)$. Let the fundamental group of C be generated by $A_1, \ldots, A_g$, $B_1, \ldots, B_g$ satisfying the relation $A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = 1$. We define a representation $\rho: \pi_1(C) \to \mathrm{SU}(r)$ by $\rho(A_1) = \alpha$, $\rho(B_1) = \beta$, $\rho(A_2) = \beta$, $\rho(B_2) = \alpha$ and $\rho(A_i) = Id_r$ if $i \neq 1, 2$. It is clear that the image of ρ is dense in $\mathrm{SU}(r)$.

Let $V \to C$ be the holomorphic vector bundle on C defined by a representation ρ as in Lemma 3.2. Then V is stable with trivial determinant and any bundle associated to V by an irreducible representation of $\mathrm{SL}(r, \mathbb{C})$ is also stable (if $\varphi: \mathrm{SL}(r, \mathbb{C}) \to \mathrm{SL}(V)$ is an irreducible representation then $\varphi\rho: \pi_1(C) \to \mathrm{SL}(V)$ is an irreducible representation of $\pi_1(C)$).

Let $X = \mathbb{P}(V)$ be the projective bundle of V and $L = \mathcal{O}_{\mathbb{P}(V)}(1)$ the relatively ample line bundle on $X = \mathbb{P}(V)$ (see Theorem 6.1).

4. The incidence variety

Let $\mathrm{Gr}(\ell, V)$ be the Grassmann bundle of ℓ-dimensional quotients of V. $\mathrm{Gr}(\ell, V)$ can also be thought of as the "space" of $(\ell - 1)$ dimensional planes in the fibres of $\pi: \mathbb{P}(V) \to C$. Let $\mathrm{Gr}(\ell, V) \times_C \mathbb{P}(V)$ be the fibre product of the Grassmann bundle and the projective bundle over C. There is a relative incidence variety Γ_ℓ in this fibre product which may be defined as follows.

Suppose

$$0 \to S \to p^* V \to Q \to 0$$

is the universal exact sequence on $\mathrm{Gr}(\ell, V)$. The fibre product $\mathrm{Gr}(\ell, V) \times_C \mathbb{P}(V)$ is the projective bundle $\mathbb{P}(p^* V)$ on $\mathrm{Gr}(\ell, V)$. There is an inclusion

$$\mathbb{P}(Q) \subset \mathbb{P}(p^* V) = \mathrm{Gr}(\ell, V) \times_C \mathbb{P}(V).$$

We define $\Gamma_\ell = \mathbb{P}(Q)$.

(Restricted to the fibres of $\mathrm{Gr}(\ell, V) \times_C \mathbb{P}(V) \to C$, Γ_ℓ is the incidence variety $\{(H, x)$ in $\mathrm{Gr} \times \mathbb{P}^{r-1}$ such that x is in $\mathrm{H}\}$.)

We denote the cohomology class determined by Γ_ℓ in $H^*(\mathrm{Gr}(\ell, V) \times_C \mathbb{P}(V), \mathbb{Z})$ by $\mathrm{d}(\Gamma_\ell)$. We can see that $\mathrm{d}(\Gamma_\ell)$ is in $H^{2(r-\ell)}(\mathrm{Gr}(\ell, V) \times_C \mathbb{P}(V), \mathbb{Z})$. We need the following lemma for computing $\mathrm{d}(\Gamma_\ell)$.

Lemma 4.1. *Let B be a complex projective manifold. Let*

$$0 \to S \to V \to Q \to 0$$

be an exact sequence of complex vector bundles on B. Let $\tau = c_1(\mathcal{O}_{\mathbb{P}(V)}(1))$ in $H^*(\mathbb{P}(V), \mathbb{Z})$. Let rank $V = r$, rank $Q = \ell$. The cohomology class of $\mathbb{P}(Q) \subset \mathbb{P}(V)$ in $H^{2(r-\ell)}(\mathbb{P}(V), \mathbb{Z})$ has the expression

$$\mathrm{d}(\mathbb{P}(Q)) = \tau^{r-\ell} - \pi^* c_1(S)\tau^{r-\ell-1} + \cdots + (-1)^{r-\ell}\pi^* c_{r-\ell}(S)$$

where π is the projection $\pi: \mathbb{P}(V) \to B$.

Proof. To determine the cohomology class of $\mathbb{P}(Q)$ as a subspace of $\mathbb{P}(V)$ it is enough to work in the C^∞ category. So we may assume that $V = S \oplus Q$. Then it can be seen that $\mathrm{d}(\mathbb{P}(Q))$ lies in the kernel of the map $H^{2(r-\ell)}(\mathbb{P}(V), \mathbb{Z}) \to H^{2(r-\ell)}(\mathbb{P}(S), \mathbb{Z})$ induced by the (C^∞) inclusion $\mathbb{P}(S) \subset \mathbb{P}(V)$. Since the only relation in $H^*(\mathbb{P}(S), \mathbb{Z})$ is

$$\tau^{r-\ell} - c_1(S)\tau^{r-\ell-1} + \cdots + (-1)^{r-\ell} c_{r-\ell}(S) = 0$$

and the inclusion $\mathbb{P}(S) \subset \mathbb{P}(V)$ induces an inclusion of complex projective spaces restricted to the fibres of $\pi: \mathbb{P}(V) \to B$, we see that

$$\mathrm{d}(\mathbb{P}(Q)) = \tau^{r-\ell} - \pi^* c_1(S)\tau^{r-\ell-1} + \cdots + (-1)^{r-\ell}\pi^* c_{r-\ell}(S).$$

COROLLARY TO LEMMA 4.1.

We have,

$$\mathrm{d}(\Gamma_\ell) = \tau_{p^*V}^{r-\ell} + c_1(Q)\tau_{p^*V}^{r-\ell-1} + \cdots$$

where

$$\tau_{p^*V} = c_1(\mathcal{O}_{\mathbb{P}(p^*V)}(1)), \text{ in } H^*(\mathrm{Gr}(\ell, V) \times_C \mathbb{P}(V), \mathbb{Z}).$$

Proof. We find from the lemma that

$$\mathrm{d}(\Gamma_\ell) = \tau_{p^*V}^{r-\ell} - c_1(S)\tau_{p^*V}^{r-\ell-1} + \cdots$$

Since $c_1(V) = 0$ for our representation bundle V, we have $c_1(Q) = -c_1(S)$. Hence the corollary.

5. The Chow divisor

Let Y be an irreducible subvariety of $\mathbb{P}(V)$ of dimension s. Then Y defines an element $\mathrm{d}(Y)$ of $H^{2r-2s}(\mathbb{P}(V), \mathbb{Z})$ and by Lemma 2.1,

$$\mathrm{d}(Y) = \pi^*\alpha\tau^{r-s} + \pi^*\beta\tau^{r-s-1}$$

where α, β are elements of $H^*(C, \mathbb{Z})$. Let $\pi | Y\colon Y \to C$ be surjective. We then have

Lemma 5.1. *If $\pi|Y: Y \to C$ is surjective, then the "relative degree" $\alpha = \pi_*(\mathrm{d}(Y) \cdot \tau^{s-1})$ in $H^0(C, \mathbb{Z})$ is a positive integer.*

Proof. Let F be a fibre $\pi^{-1}(x)$ of π: $\mathbb{P}(V)\to C$ for some x in C. Then $F\cong\mathbb{P}^{r-1}$ and $Y\cap F$ is a subvariety of dimension $(s-1)$ of $F\cong\mathbb{P}^{r-1}$ (this is because (i) $Y\cap F$ is nonempty by hypothesis and (ii) any irreducible family over a curve is flat and hence the fibre dimension is a constant). The degree of $Y\cap F$ in $F\cong\mathbb{P}^{r-1}$ is the cohomology class $\mathrm{d}(Y\cap F)\cdot\mathrm{d}(\mathcal{O}_{\mathbb{P}^{r-1}}(1))^{s-1}$. By Lemma 2.1 we see that q: $H^{2r-2}(\mathbb{P}(V),\mathbb{Z})\to H^{2r-2}(F,\mathbb{Z})$ is a surjection, and τ^{r-1} maps to the generator of $H^{2r-2}(F,\mathbb{Z})$ $(=c_1(\mathcal{O}_{\mathbb{P}^{r-1}}(1))^{r-1})$ under this map. We obtain

$$\begin{aligned}\text{degree}\,(Y\cap F)&=q(\mathrm{d}(Y)\cdot\tau^{s-1})\\&=q(\pi^*\alpha\cdot\tau^{r-1}+\pi^*\beta\tau^{r-2}).\end{aligned}$$

Again, by Lemma 2.1,

$$H^{2r-2}(\mathbb{P}(V),\mathbb{Z})\cong H^0(C,\mathbb{Z})\cdot\tau^{r-1}\oplus H^2(C,\mathbb{Z})\cdot\tau^{r-2}$$

and $\ker(q)=H^2(C,\mathbb{Z})\cdot\tau^{r-2}$, while q maps $H^0(C,\mathbb{Z})\cdot\tau^{r-1}$ isomorphically onto $H^{2r-2}(F,\mathbb{Z})$. Therefore

$$\deg(Y\cap F)=\pi_*(\pi^*\alpha\cdot\tau^{r-1})=\alpha$$

α represents the relative degree and is a positive integer.

We will now associate to a Y as above, a divisor D_Y in $\mathrm{Gr}(r-s,V)$ as follows.

Let p_1 and p_2 be the two projections from $\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)$ to $\mathrm{Gr}(r-s,V)$ and $\mathbb{P}(V)$ respectively. By definition, $(p_2|\Gamma_{r-s})^{-1}(Y)$ is the fibre product

$$\begin{array}{ccc}(p_2|\Gamma_{r-s})^{-1}(Y)&\subset&\Gamma_{r-s}\\ \Big\downarrow{\scriptstyle p_2}&&\Big\downarrow{\scriptstyle (p_2|\Gamma_{r-s})}\\ Y&\subset&\mathbb{P}(V)\end{array}$$

We have,

Lemma 5.2. *Let Y be a reduced and irreducible subvariety of $\mathbb{P}(V)$ of dimension s such that $\pi|Y\colon Y\to C$ is surjective. Then we have*

(i) *$(p_2|\Gamma_{r-s})^{-1}(Y)$ is a reduced and irreducible subvariety of Γ_{r-s}, and it is flat over C. Also, dimension $((p_2|\Gamma_{r-s})^{-1}(Y))=s(r-s)$.*

(ii) *In the integral cohomology ring of* $\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)$, *we have the following equality*

$$\mathrm{d}((p_2|\Gamma_{r-s})^{-1}(Y))=p_2^*(\mathrm{d}(Y))\cdot\mathrm{d}(\Gamma_{r-s})$$

where $p_2^*\colon H^*(\mathbb{P}(V),\mathbb{Z})\to H^*(\mathrm{Gr}(r-s)\times_C\mathbb{P}(V),\mathbb{Z})$ is the pull back map on the cohomology.

Proof. (i) From the definition of $(p_2|\Gamma_{r-s})^{-1}(Y)$ as a fibre product, we see that since $(p_2|\Gamma_{r-s})$ is a smooth morphism, so is $\tilde{p}_2$. Since Y is reduced and irreducible, the smoothness of $\tilde{p}_2$ implies that $(p_2|\Gamma_{r-s})^{-1}(Y)$ is reduced and irreducible. It is clear that $(\pi\tilde{p}_2)\colon(p_2|\Gamma_{r-s})^{-1}(Y)\to C$ is surjective, and hence flat.

Since $(p_2|\Gamma_{r-s})\colon\Gamma_{r-s}\to\mathbb{P}(V)$ is flat (in fact, smooth), we see by base change that $\tilde{p}_2$:

$(p_2|\Gamma_{r-s})^{-1}(Y)\to Y$ is flat. Therefore $\tilde{p}_2$ is equidimensional, and it is enough to compute the fibre dimension of $\tilde{p}_2$ in order to compute the required dimension. The fibre dimension of p_2 is the dimension of the space of $(r-s-1)$ dimensional planes passing through a fixed point of $\mathbb{P}^{r-1}$, namely $s(r-s-1)=s(r-s)-s$. Therefore, dimension $((p_2|\Gamma_{r-s})^{-1}(Y))=s(r-s)$.

(ii) We remark that $(p_2|\Gamma_{r-s})^{-1}(Y)$ as a subvariety of $\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)$ is also $p_2^{-1}(Y)\cap\Gamma_{r-s}$, where $p_2^{-1}(Y)$ is defined by the fibre product

$$\begin{array}{ccc} p_2^{-1}(Y) & \subset & \mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V) \\ \downarrow & & \downarrow{\scriptstyle p_2} \\ Y & \subset & \mathbb{P}(V) \end{array}$$

We can check that

$$\begin{aligned} &\dim(\Gamma_{r-s})+\dim(p_2^{-1}(Y))-\dim(\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)) \\ &=s(r-s) \\ &=\dim((p_2|\Gamma_{r-s})^{-1}(Y)=p_2^{-1}(Y)\cap\Gamma_{r-s}) \end{aligned}$$

where the last equality of dimensions is by the earlier part of this lemma.

This shows that $p_2^{-1}(Y)$ and Γ_{r-s} intersect properly in $\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)$, and therefore, by results from intersection theory (see Appendix A, axiom A6 and Theorem (1.1), in [1])

$$\mathrm{d}((p_2|\Gamma_{r-s})^{-1}(Y))=p_2^*(\mathrm{d}(Y))\cdot\mathrm{d}(\Gamma_{r-s})$$

in the Chow ring of $\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)$. But we observe that the ring homomorphism $\mathrm{CH}^*(X)\to H^*(X)$ from the Chow ring to the singular cohomology ring preserves intersection of cycles for any nonsingular variety X, and therefore we have the required expression in the cohomology of $\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)$.

We now define

$$D_Y=p_1((p_2|\Gamma_{r-s})^{-1}(Y))$$

as the image (scheme-theoretic) of $(p_2|\Gamma_{r-s})^{-1}(Y)$. We will consider D_Y as a subvariety of $\mathrm{Gr}(r-s,V)$ in this sense. We now have

Lemma 5.3. *Let $Y\subset\mathbb{P}(V)$ be reduced and irreducible of dimension s, such that $\pi|Y$: $Y\to C$ is surjective. Then the variety D_Y defined above satisfies*

(i) *D_Y is a reduced and irreducible subvariety of* $\mathrm{Gr}(r-s,V)$

(ii) *D_Y is a divisor in* $\mathrm{Gr}(r-s,V)$. *The morphism $p:D_Y\to C$ is surjective (and flat).*

Proof. (i) D_Y is the image of the reduced and irreducible variety $(p_2|\Gamma_{r-s})^{-1}(Y)$ (see Lemma 5.2 (i)) under the morphism p_1, and is hence reduced and irreducible.

(ii) Since D_Y is reduced and irreducible, it is enough to compute the dimension of D_Y at a nonsingular point of D_Y to compute the dimension of D_Y.

Let F be a fibre of $\pi:\mathbb{P}(V)\to C$ so that F is isomorphic to a $(r-1)$ dimensional projective space. We consider the $(s-1)$ dimensional variety $Y\cap F\subset F$ (by flatness,

all fibres of $Y \to C$ are $(s-1)$ dimensional). Through a nonsingular point of $Y \cap F$, there exists a $(r-s-1)$ dimensional linear subspace L of F which intersects $(Y \cap F)$ at only finitely many points. This is clear because there is a $(r-s)$ dimensional linear subspace of F which meets $(Y \cap F)$ in finitely many points (viz., the degree of $(Y \cap F)$ in F) and we can obtain a $(r-s-1)$ dimensional linear subspace passing through one of these points. This $(r-s-1)$ linear subspace L defines a point of D_Y, with $p(L) = \pi(F)$ in C (p is the projection $p\colon \mathrm{Gr}(r-s, V) \to C$). From the diagram

$$\begin{tikzcd}[column sep=small]
 & & (p_2|\Gamma_{r-s})^{-1}(Y) \arrow[dll, "p_2"'] \arrow[drr, "p_1"] & & \\
Y & & \cap & & D_Y \\
 & & \Gamma_{r-s} \arrow[dll, "p_2"'] \arrow[drr, "p_1"] & & \\
\mathbb{P}(V) & \cap & & \cap & \mathrm{Gr}(r-s, V)
\end{tikzcd}$$

it is clear that the fibre of p_1 over this point L of D_Y is the set

$$\{(L, x) \text{ in } \mathrm{Gr}(r-s, V) \times_C \mathbb{P}(V) \text{ such that } x \text{ is in } (Y \cap F \cap L)\}.$$

This shows that the dimension of D_Y at this point L is the dimension of $(p_2|\Gamma_{r-s})^{-1}(Y)$ which is $s(r-s)$ (by Lemma (5.2) (i)). Hence the dimension of D_Y is $s(r-s)$ everywhere, and therefore D_Y is a divisor in $\mathrm{Gr}(r-s, V)$ (which is of dimension $s(r-s)+1$).

The commutative diagram

$$\begin{tikzcd}
(p_2|\Gamma_{r-s})^{-1}(Y) \arrow[d, "(\pi \tilde{p}_2)"'] \arrow[dr, "p_1"] & \\
C & D_Y \arrow[l, "p"]
\end{tikzcd}$$

shows that $p\colon D_Y \to C$ is surjective (from Lemma 5.2 (i)) and is hence flat, since D_Y is irreducible and C is a curve.

Lemma 5.4. Let $Y \subset \mathbb{P}(V)$ be an irreducible and reduced subvariety of $\mathbb{P}(V)$ of dimension s, such that $\pi|Y\colon Y \to C$ is surjective. Let

$$\mathrm{d}(Y) = \pi^*\alpha\tau^{r-s} + \pi^*\beta\tau^{r-s-1}$$

be the class of Y in the cohomology ring $H^{2r-2s}(\mathbb{P}(V), \mathbb{Z})$, where $\alpha \in H^0(C, \mathbb{Z})$ and $\beta \in H^2(C, \mathbb{Z})$. Then the divisor D_Y in $Gr(r-s, V)$ (see Lemma 5.3) has the expression

$$\delta \mathrm{d}(D_Y) = (p^*\alpha)c_1(Q) + p^*(\beta)$$

in $H^2(Gr(r-s, V), \mathbb{Z})$, where Q is the universal quotient bundle on $Gr(r-s, V)$ (see §2), and δ is the degree of the extension of function fields defined by $p_1\colon (p_2|\Gamma_{r-s})^{-1}(Y) \to D_Y$ (δ is finite by Lemma 5.2 (i) and Lemma 5.3 (ii)).

Proof. From the commutative diagram

$$\begin{tikzcd}[column sep=small] & \mathrm{Gr}(r-s,\pi^*V)=\mathrm{Gr}(r-s,V)\times_C\mathbb{P}(V)=\mathbb{P}(p^*V) \arrow[dl, "p_2"'] \arrow[dr, "p_1"] & \\ \mathbb{P}(V) \arrow[dr, "\pi"'] & & \mathrm{Gr}(r-s,V) \arrow[dl, "p"] \\ & C & \end{tikzcd}$$

it follows that $p_{1*}, p_2^*\pi^* = p^*$ and $p_2^*(c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) = c_1(\mathcal{O}_{\mathbb{P}(p^*V)}(1))$. Also,

$$\begin{aligned}\delta\mathrm{d}(D_Y) &= p_{1*}((p_2|\Gamma_{r-s})^{-1}(Y)) \text{ (see definition of } p_{1*} \text{ in Appendix A of [1])}\\ &= p_{1*}(p_2^*(\mathrm{d}(Y))\cdot\mathrm{d}(\Gamma)) \text{ (by Lemma 5.2 (ii))}\\ &= p_{1*}(p_2^*(\pi^*\alpha\tau^{r-s}+\pi^*\beta\tau^{r-s-1})\cdot(\tau^s_{p^*V}+p_1^*c_1(Q)\tau^{s-1}_{p^*V}+\cdots)\\ &\quad \text{(by Corollary to Lemma 4.1)}\\ &= p_{1*}((p_2^*\pi^*\alpha)(p_1^*c_1(Q))\tau^{r-1}_{p^*V}+(p_2^*\pi^*\beta)\tau^{r-1}_{p^*V})\\ &= (p^*\alpha)c_1(Q)+p^*\beta.\end{aligned}$$

6. The example

We can now prove the following.

Theorem 6.1. *Let $X=\mathbb{P}(V)$ be the ruled variety (of dimension $r=\operatorname{rank} V$) and L the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1)$ on X, where V is a vector bundle on the compact Riemann surface C defined by a special unitary representation of the fundamental group $\pi_1(C)$ whose image is dense in* SU(r) *(see §3). Then L is ample when restricted to a proper subvariety of X, but L is not ample on X.*

Proof. Let Y be a reduced and irreducible subvariety of $\mathbb{P}(V)$. If Y is fibre of $\pi\colon\mathbb{P}(V)\to C$, then $L|Y$ is clearly ample on Y. Therefore let $\pi|Y\colon Y\to C$ be surjective, and let $\dim Y=s$. The notation will be as in Lemma 5.4.

We observe that every line bundle on $\mathrm{Gr}(r-s,V)$ is of the form $(\det Q)^{\otimes m}\otimes p^*M$ for some integer m, and line bundle M on C. In particular, so is the line bundle defined by the effective divisor δD_Y in $\mathrm{Gr}(r-s,V)$. On the other hand, $c_1((\det Q)^{\otimes m}\otimes p^*M)=mc_1(Q)+p^*c_1(M)$ and if this line bundle is to be defined by δD_Y, then $m=\alpha$ and $c_1(M)=\beta$ (see Lemma 5.4). By Lemma 5.1, $\alpha>0$. We have,

$$H^0(\mathrm{Gr}(r-s,V),(\det Q)^{\otimes\alpha}\otimes p^*M)=H^0(C,p_*((\det Q)^{\otimes\alpha})\otimes M).$$

Since δD_Y defines a nontrivial section of $(\det Q)^{\otimes\alpha}\otimes p^*M$, we have

$$H^0(C,p_*((\det Q)^{\otimes\alpha})\otimes M)\neq 0.$$

Therefore there is a nontrivial homomorphism

$$M^*\to p_*((\det Q)^{\otimes\alpha}).$$

The bundle $p_*((\det Q)^{\otimes\alpha})$ is associated to V by the irreducible representation of $\mathrm{GL}(r, \mathbb{C})$ on $H^0(\mathrm{Gr}(r-s, r), (\det Q)^{\otimes\alpha})$ and therefore it is a stable bundle (since V is defined by a dense $\mathrm{SU}(r)$ representation, $p_*((\det Q)^{\otimes\alpha})$ is also defined by an irreducible special unitary representation). Therefore we obtain

$$-\beta = c_1(M^*) < \mu(p_*((\det Q)^{\otimes\alpha})) = 0.$$

On the other hand,

$$\begin{aligned}\tau^s\cdot \mathrm{d}(Y) &= \tau^s\cdot(\pi^*\alpha\tau^{r-s} + \pi^*\beta\tau^{r-s-1})\\ &= \beta\end{aligned}$$

and therefore, $\tau^s\cdot \mathrm{d}(Y) > 0$.

An application of Nakai's criterion now shows that if Z is any proper subvariety of $X = \mathbb{P}(V)$, then $L|Z$ is ample on Z. Also, $c_1(L)^r = \tau^r = 0$, and hence L is not ample on X. This completes the proof.

7. Hartshorne's theorem

A stable vector bundle of positive degree on a compact Riemann surface is ample by a result of Hartshorne (cf. [3]). The above arguments can be seen to give a proof of this result. Let V be a stable vector of positive degree on a compact Riemann surface C. We will show that $\mathcal{O}_{\mathbb{P}(V)}(1)$ is ample. Let Y be an integral (reduced and irreducible) subvariety of $\mathbb{P}(V)$ of dimension s, $0 < s < r = \operatorname{rank} V$. If Y is contained in a fibre of $\pi\colon \mathbb{P}(V) \to C$, then $\mathcal{O}_{\mathbb{P}(V)}(1)|\,Y$ is clearly ample. Let $\pi|\,Y\colon Y \to C$ be a surjective morphism, and let D_Y be the associated divisor in $\mathrm{Gr}(r-s, V)$ (see Lemma 5.3). We consider the nonzero homomorphism

$$M^* \to p_*((\det Q)^{\otimes\alpha})$$

obtained as in the proof of Theorem 6.1. (The notation is as in Lemma 5.4). Since $p_*((\det Q)^{\otimes\alpha})$ is associated to the stable bundle V by an irreducible representation, it is semistable. Therefore

$$\mu(M^*) = -\beta \leqslant \mu(p_*((\det Q)^{\otimes\alpha})).$$

Also,

$$\mu(p_*((\det Q)^{\otimes\alpha})) = \alpha(r-s)\mu(V) > 0.$$

Therefore,

$$\alpha(r-s)\mu(V) + \beta \geqslant 0.$$

On the other hand,

$$\alpha c_1(V) > \alpha(r-s)(c_1(V)/r) = \alpha(r-s)\mu(V)$$

and hence

$$\alpha c_1(V) + \beta > 0.$$

Since $\tau^s\cdot \mathrm{d}(Y) = \alpha c_1(V) + \beta$, and since $\tau^r = c_1(V) > 0$, we see by Nakai's criterion that $\mathcal{O}_{\mathbb{P}(V)}(1)$ is ample.

8. Bogomolov stability and the restriction theorem

Let $W \to X$ be a holomorphic vector bundle of rank r on a complex projective manifold X. Then W is Bogomolov stable if for any irreducible representation $\rho: \mathrm{GL}(r, \mathbb{C}) \to \mathrm{SL}(n, \mathbb{C})$, we have $H^0(X, W_\rho) = 0$, where W_ρ is the vector bundle associated to W by the representation ρ. When the rank r is two, every irreducible representation of the standard two dimensional representation $\mathrm{GL}(V)$ is of the form $S^{2m}(V) \otimes (\det V)^{-m}$ for a positive integer m. Therefore, a rank two bundle W is Bogomolov stable if and only if $H^0(X, S^{2m}(W) \otimes (\det W)^{-m}) = 0$ for all $m > 0$. The following example (due to a discussion with V Mehta) shows that the restriction theorem (the analogue of the result of Mehta and Ramanathan for Mumford stability) is false for Bogomolov stability.

Let $X = \mathbb{P}(V)$ as in Theorem 6.1 and $L = \mathcal{O}_{\mathbb{P}(V)}(1)$. Let

$$0 \to L \to W \to \mathcal{O} \to 0$$

be a nonsplit extension on X defined by a nonzero element of $H^1(X, L)$ (we observe that $h^1(X, L) = h^1(C, V) = h^0(C, V^* \times K_C) \geqslant \mathrm{rank}(V)(g-1) > 0$ with notation as in Theorem 6.1 and Section 3). We need the following simple lemma.

Lemma 8.1. *Let W be a two dimensional vector space and*

$$0 \to L_1 \to W \to L_2 \to 0$$

a filtration by one dimensional vector spaces L_1, L_2. We then have

(i) *There is a canonical filtration on $S^i W$,*

$$0 \subset F_0 \subset F_1 \subset \cdots \subset F_i = S^i W$$

with $F_j = L_1^{\otimes(i-j)} \otimes S^j W$, and

$$\begin{aligned} F_j/F_{j-1} &= L_1^{\otimes(i-j)} \otimes S^j W / L_1^{\otimes(i-j+1)} \otimes S^{j-1} W \\ &= L_1^{\otimes(i-j)} \otimes L_2^{\otimes j} \end{aligned}$$

(ii) *There is a canonical commutative diagram*

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ & & S^{i-2}W \otimes L_1^2 & = & S^{i-2}W \otimes L_1^2 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \to & S^{i-1}W \otimes L_1 & \to & S^i W & \to & L_2^i \to 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \to & L_1 \otimes L_2^{i-1} & \to & W \otimes L_2^{i-1} & \to & L_2^i \to 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

Proof. We omit the proof which is elementary algebra.

The exact sequence defining the vector bundle W induces the following filtration on $S^{2m}(W) \otimes (\det W)^{-m}$

$$0 \subset \tilde{F}_0 \subset \tilde{F}_1 \subset \cdots \subset \tilde{F}_{2m} = S^{2m}(W) \otimes (\det W)^{-m}$$

with $\tilde{F}_i = S^i(W)\otimes L^{\otimes(m-i)}$, $0\leqslant i\leqslant 2m$, and we have

$$0\to \tilde{F}_{i-1}\to \tilde{F}_i \to L^{\otimes(m-i)}\to 0$$

(this follows from (i) of Lemma (8.1) above). From (ii) of Lemma (8.1) above, we get a commutative diagram of vector bundles

$$\begin{array}{ccccccccc} 0\to & \tilde{F}_{i-1} & \to & \tilde{F}_i & \to & L^{\otimes(m-i)} & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0\to & L^{\otimes(m-i+1)} & \to & W\otimes L^{\otimes(m-i)} & \to & L^{\otimes(m-i)} & \to 0 \\ & \downarrow & & \downarrow & & & \\ & 0 & & 0 & & & \end{array}$$

and the long exact cohomology sequence associated to this diagrams shows that the element of $H^1(X,\tilde{F}_{i-1}\otimes L^{(m-i)})$ defining $\tilde{F}_i$ as an extension of $\tilde{F}_{i-1}$ by $L^{\otimes(m-i)}$ maps to the element of $H^1(X, L^{(m-i+1)-(m-i)}) = H^1(X,L)$ defining W as an extension of L by $\mathcal{O}$. This shows that

$$(*) \qquad 0\to \tilde{F}_{i-1}\to \tilde{F}_i\to L^{\otimes(m-i)}\to 0$$

is a nonsplit extension for $1\leqslant i\leqslant 2m$.

Now, if $\mathcal{O}\to S^{2m}(W)\otimes(\det W)^{-m}$ is a nonzero homomorphism, then it factors by $\mathcal{O}\to \tilde{F}_m\subset S^{2m}(W)\otimes(\det W)^{-m}$ and $\text{image}(\mathcal{O})\cap \tilde{F}_{m-1}=0$, since $H^0(X,L^{\otimes i})=0$ if $i\neq 0$ (see Theorem 6.1). This also shows that $\mathcal{O}\to \tilde{F}_m$ defines a splitting of

$$0\to \tilde{F}_{m-1}\to \tilde{F}_m\to \mathcal{O}\to 0$$

contradicting that (*) is a nonsplit extension. Therefore $H^0(X, S^{2m}W\otimes(\det W)^{-m})=0$ and W is Bogomolov stable on X.

On the other hand, if Y is any proper subvariety of X (in particular, a hyper surface) then $W|Y$ is defined by the extension

$$0\to L|Y\to W|Y\to \mathcal{O}_Y\to 0$$

and since $L|Y$ is ample on Y, we see that for large m,

$$H^0(Y, S^{2m}(W|Y)\otimes(L|Y)^{-m})\supset H^0(Y,(L|Y)^{\otimes m})\neq 0$$

and $W|Y$ is Bogomolov instable.

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References

[1] Hartshorne R, *Algebraic Geometry* (New York, Heidelberg, Berlin: Springer-Verlag) (1977)
[2] Hartshorne R, Ample subvarieties of algebraic varieties, *Lecture Notes No.* 156 (Springer-Verlag) (1970)

[3] Hartshorne R, Ample vector bundles on curves, *Nagoya Math. J.* **43** (1971) 73–89
[4] Husemoller D, *Fibre Bundles* (New York, Heidelberg, Berlin: Springer-Verlag) (1974)
[5] Narasimhan M S and Seshadri C S, Stable and unitary vector bundles on compact Riemann surfaces. *Ann. Math.* **82**, (1965) 540–567

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 3, December 1989, pp. 209–215.

Graded subrings of $\mathbb{C}[X, Y]$

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Abstract. We prove the following result

Theorem. *Let R be an affine normal 2-dimensional subring of $\mathbb{C}[X, Y]$ generated by homogeneous polynomials, say, $F_1, \ldots, F_n$. If the g.c.d. $(F_1, \ldots, F_n)$ has at most two distinct linear factors (possibly occurring with multiplicities), then R is isomorphic to a ring of invariants $\mathbb{C}[v, V]^W$ for some finite group W.*

This generalizes a result of D Anderson on rings generated by monomials. As a corollary of the theorem, we prove some special cases of a conjecture of CTC wall. We also prove some results on general graded affine subrings of $\mathbb{C}[X, Y]$.

Keywords. Affine normal subrings of $\mathbb{C}[X, Y]$; ring of invariants; graded affine subrings.

Introduction

The main motivation for this work is the following conjecture of C T C Wall.

Conjecture. Let G be a reductive algebraic group acting linearly on a complex vector space V with $\dim V/G = 2$. Then V/G is isomorphic to a quotient $\mathbb{C}^2/W$ of $\mathbb{C}^2$ by a finite group W. See [10].

Wall has remarked that the analogous result is not true for higher-dimensional quotients. For an easy example, see §3. In [10] Wall has given a proof of the conjecture when G is an algebraic torus. While attempting to prove the above conjecture, it was hoped that the following more general question has an affirmative answer.

(*) *Question.* Let $\mathbb{C} \subset R \subset \mathbb{C}[X, Y]$, where R is a 2-dimensional affine normal subring of $\mathbb{C}[X, Y]$ generated by homogeneous polynomials. Suppose R has a rational singularity. Is R isomorphic to a ring of invariants $\mathbb{C}[U, V]^W$ for some finite group W?

The answer to (*) turns out to be negative in general, but we have some affirmative results. The main result in this paper is the following.

Theorem (§2) *With R as above, let $F_1, \ldots, F_n$ be some homogeneous generators of R. If the g.c.d. $(F_1, \ldots, F_n)$ has at most two distinct linear factors (possibly occurring with multiplicities), then the answer to (*) is affirmative. In fact, if R is birational with $\mathbb{C}[X, Y]$, then the condition on g.c.d. (having at most 2 distinct linear factors) is automatically satisfied. Further, if R is birational with $\mathbb{C}[X, Y]$, then the group W can be assumed to be cyclic (but not in general).*

In [1] Anderson studied the case when R is generated by monomials and proved

that (*) is true in this case. In [5] Huang has studied rings generated by monomials in >2 variables. Our theorem above is thus more general than Anderson's result. The methods in [1] are algebraic whereas our method is algebraic geometric. Some of the key results used in the proof of the above theorem are results of Gurjar and Shastri on varieties dominated by $\mathbb{C}^2$ see [3,4]. The Proposition 2 in this paper is of independent interest. In §2, Remark 2, we explain the connection between (*) and Wall's conjecture. In §3, we give some examples. In particular, we shall see that the co-ordinate ring of a quasi-homogeneous normal affine surface with a rational singularity at the vertex is a graded subring of $\mathbb{C}[X, Y]$. This uses essentially the results of Pinkham in [8].

2. A special case of (*)

All the varieties considered will be defined over $\mathbb{C}$. We begin with some preliminaries.

Lemma 1. *Let $\mathbb{C} \subset R \subset \mathbb{C}[X_1,\dots,X_n]$, where R is an affine ring generated by homogeneous polynomials in the indeterminates $X_1,\dots,X_n$. Then the integral closure of R in its quotient field (or in $\mathbb{C}(X_1,\dots,X_n)$) is also generated by homogeneous polynomials. If R is generated by monomials, so is the integral closure.*

Proof. We can refer to Bourbaki [2, chapter V, §1.8]. For the sake of completeness, we briefly indicate a proof when R is generated by monomials. Consider the action of torus $T=\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ on $\mathbb{C}[X_1,\dots,X_n]$ given by $(t_1,\dots,t_n)\circ(X_1,\dots,X_n)=(t_1X_1,\dots,t_nX_n)$. Then R is invariant under this action.

Let φ be a polynomial in $X_1,\dots,X_n$ integral over R. Write $\varphi=\varphi_d+\varphi_{d+1}+\cdots$ where φ_i are homogeneous components of φ. φ satisfies an integral equation $\varphi^l+a_1\cdot\varphi^{l-1}+\cdots+a_l=0$ with $a_i \in R$. Using the T-action, we see that $(t_1,\dots,t_n)\circ\varphi$ is also integral over R for every $(t_1,\dots,t_n)\in T$. From this it is easy to see that $\varphi_d, \varphi_{d+1},\dots$ are integral over R.

Lemma 2. *Let $\mathbb{C} \subset R \subset \mathbb{C}[X, Y]$ where R is an affine normal subring of $\mathbb{C}[X, Y]$ generated by homogeneous polynomials and having same quotient field as $\mathbb{C}[X, Y]$. Let W be the affine variety corresponding to R and $\pi\colon \mathbb{C}^2 \to W$ the natural dominant morphism. Then W has at most one singular point; say p. If W is smooth, then $W \approx \mathbb{C}^2$. Otherwise, $\mathbb{C}^2-\pi^{-1}(p)\to W-\{p\}$ is an open embedding.*

Proof. The first two assertions are easy, so we prove only the last assertion.

Let $R=\mathbb{C}[\mathrm{HG}_1,\dots,\mathrm{HG}_n]$, where HG_i are homogeneous polynomials with g.c.d. H. Then $\pi^{-1}(p)=\{H=0\}$, which is a union of lines through $(0,0)$ in $\mathbb{C}^2$. Since π is birational, there exist only finitely many points $p=p_1,\dots,p_l$ such that $\pi^{-1}(p_i)$ is positive dimensional.

Let $\deg \mathrm{HG}_i=d_i$ and $p_2=(\lambda_1,\dots,\lambda_n)$ (in the natural embedding of W in $\mathbb{C}^n$) if $l \geqslant 2$. Then $\lambda_i \neq 0$ for some i. W has an obvious $\mathbb{C}^*$ action, under which $(t^{d_1}\lambda_1,\dots,t^{d_n}\lambda_n)\in W$ for $t\in\mathbb{C}^*$. For any $q\in\mathbb{C}^2$ with $\pi(q)=p_2$, we have $\pi((t)q)=(t^{d_1}\lambda_1,\dots,t^{d_n}\lambda_n)$. Thus $\pi^{-1}((t^{d_1}\lambda_1,\dots,t^{d_n}\lambda_n))$ is also positive dimensional. This is not possible, hence $l=1$. Now $\mathbb{C}^2-\pi^{-1}(p)\subset W-\{p\}$ by Zariski's main theorem. The image of $L-\{(0,0)\}$ under π is the orbit of a point of W under the $\mathbb{C}^*$-action, where L is any line in $\mathbb{C}^2$ through the origin.

Lemma 3. *Let $\mathbb{C} \subset R \subset \mathbb{C}[X, Y]$, where R is generated by homogeneous polynomials $F_1, \ldots, F_n$. If $F_1, \ldots, F_n$ have no nontrivial common factor and $\dim R = 2$ then $\mathbb{C}[X, Y]$ is integral over R.*

Proof. Let W be the affine variety corresponding to R and $\pi: \mathbb{C}^2 \to W$ as in Lemma 2. It suffices to prove that π is a proper map. For any $(\alpha, \beta) \in \mathbb{C}^2$ with $(\alpha, \beta) \neq (0, 0)$, at least one of the co-ordinates $F_i(\alpha, \beta) \neq 0$. Let S^3 be the unit sphere in $\mathbb{C}^2$. Then $\pi(S^3)$ is a compact subset of W not containing $\pi((0,0))$. For any $q \neq (0,0)$ in $\mathbb{C}^2$, $\pi(q) = (\|q\|^{d_1} F_1(q/\|q\|), \ldots, \|q\|^{d_n} F_n(q/\|q\|))$. As $\|q\| \to \infty$, one of the co-ordinates of $\pi(q)$ tends to ∞.

Next we prove the main result of this paper.

Theorem. *Let $\mathbb{C} \subset R \subset \mathbb{C}[X, Y]$, where R is an affine, normal subring of dimension 2 generated by homogeneous polynomials $F_1, \ldots, F_n$. Let H be the g.c.d. of $F_1, \ldots, F_n$. Suppose H has at most two distinct linear factors (possibly occurring with multiplicities).*

Then $R \approx \mathbb{C}[U, V]^W$, where $W \subset GL(2, \mathbb{C})$ is a finite group acting linearly on the polynomial ring $\mathbb{C}[U, V]$. If R is birational with $\mathbb{C}[X, Y]$, then W can be assumed to be cyclic.

Proof. First we consider the case when H is a non-zero constant. In this case, by Lemma 3 $\mathbb{C}[X, Y]$ is integral over R. Now we invoke a result of Gurjar and Shastri, to conclude that $R \approx \mathbb{C}[U, V]^W$, see [3]. (This result was first proved by Miyanishi by a different method in [6]).

Now suppose H has one or two linear factors. By Lemma 1, the integral closure $\bar{R}$ of R in the quotient field of $\mathbb{C}(X, Y)$ is an affine, normal two-dimensional ring generated by homogeneous polynomials.

Let $S \in \bar{R}$ be a non-constant homogeneous element. Then S satisfies an equation $S^l + a_1 S^{l-1} + \cdots + a_l = 0$ with a_i non-constant homogeneous elements of R. This shows that any linear factor of H divides S as well. We will first prove that $\bar{R} \approx \mathbb{C}[U, V]^W$ with $W \subset GL(2, \mathbb{C})$ a cyclic group. Let W be the normal quasi-homogeneous affine variety corresponding to $\bar{R}$ and $\pi: \mathbb{C}^2 \to W$ as before. Letting $\tilde{H}$ the g.c.d. of the generators of $\bar{R}$, we have proved that $\tilde{H}$ has at most two distinct linear factors. Let p be the "vertex" of W. The $\mathbb{C}^*$-action on $W - \{p\}$ keeps the Zariski open subset $\mathbb{C}^2 - \pi^{-1}(p) = \mathbb{C}^2 - \{\tilde{H} = 0\}$ stable. It is well-known (see, for example [7]) that $W - \{p\}/\mathbb{C}^* \approx \mathbb{P}^1$, and $\mathbb{C}^2 - \{\tilde{H} = 0\}/\mathbb{C}^* \approx \mathbb{P}^1$-{one or two points}. It follows that the fibration $W - \{p\} \to \mathbb{P}^1$ has at most two multiple fibres. By results of Orlik-Wagreich in [7], it follows that the canonical equivariant resolution of the singularity at p has the exceptional divisor with linear dual graph. The local fundamental group at p is therefore cyclic.

It is easy to see that the fundamental group at infinity for W is isomorphic to the local fundamental group at p (using the $\mathbb{C}^*$-action on $W - \{p\}$). W is itself topologically contractible, and by what we have proved above it has finite fundamental group at infinity. Now we use the topological characterization of $\mathbb{C}^2/W$ due to Gurjar and Shastri to conclude that $\bar{R} \approx \mathbb{C}[U, V]^W$, where $W \subset GL(2, \mathbb{C})$ is isomorphic to the local fundamental group at p, see [4].

Now we deal with the general case of R. $\bar{R} \supset R$ is a finite integral extension and $\bar{R} \approx \mathbb{C}[U, V]^W$. Thus there exists a proper morphism $\mathbb{C}^2 \to V$, where V is the affine, normal variety corresponding to R. Again by the result in [3] used earlier, $R \approx \mathbb{C}[U, V]^G$ where $G \subset GL(2, \mathbb{C})$ is a finite subgroup.

Remark 1. Let $R=\mathbb{C}[X^4Y^2, X^5Y^2(X^2-Y^2), X^3Y(X^2+Y^2)]$. Using the results of Riemenschneider, one can see that $R\approx\mathbb{C}[U,V]^G$ where G is a dihedral subgroup of $GL(2,\mathbb{C})$ of order 8. One can show easily that R is not birational with $\mathbb{C}[X,Y]$, see [9] and Prop. 2.

The above theorem is a generalization of Anderson's results on rings generated by monomials, see [1].

Remark 2. We explain the connection between Wall's conjecture and the question (*) mentioned in the introduction.

Let $G\subset GL(n,\mathbb{C})$ be a reductive group acting linearly on $\mathbb{C}[X_1,\dots,X_n]$. Then $R=\mathbb{C}[X_1,\dots,X_n]^G$ is an affine, normal ring generated by homogeneous polynomials. Boutot has proved that R has rational singularities.

Now let $\dim R=2$ and $\mathbb{C}^n\xrightarrow{\pi}\mathbb{C}^n/G$ the quotient map. If we chose a general 2-dimensional linear space $L\subset\mathbb{C}^n$ passing through the origin $(0,\dots,0)$, then $\pi|_L$ is a dominant morphism. We see that R is isomorphic to a subring of $\mathbb{C}[X,Y]$ generated by homogeneous element and R has rational singularities (at most one by Lemma 2). For counter examples to (*), see §3.

Remark 3. The normal surface $W\colon\{X^n+Y^n=Z^{n+1}\}\subset\mathbb{C}^3$ is parametrized by $\bar{X}=U(U^n+V^n)$, $\bar{Y}=V(U^n+V^n)$, $\bar{Z}=U^n+V^n$. For $n\geqslant 3$, W has a non-rational singularity. Note that $\mathbb{C}[\bar{X},\bar{Y},\bar{Z}]\subset\mathbb{C}[U,V]$ has same quotient field as $\mathbb{C}[U,V]$ but for $n\geqslant 3$, the hypothesis of the Theorem is not satisfied. Also $\mathbb{C}[\bar{X},\bar{Y},\bar{Z}]$ is integral over the graded subring $\mathbb{C}[\bar{X},\bar{Y}]\approx\mathbb{C}[T_1,T_2]$.

Next we prove a partial converse of the Theorem above.

PROPOSITION 1.

Let $\mathbb{C}\subset R\subset\mathbb{C}[X,Y]$ where R is a normal, affine subring generated by homogeneous polynomials $F_1,\dots,F_n$. Suppose the quotient field of $R=\mathbb{C}(X,Y)$ and R has a rational singularity at its "vertex". Then the g.c.d. $(F_1,\dots,F_n)$ has at most two distinct linear factors. (Therefore, $R\approx\mathbb{C}[U,V]^G$ with finite cyclic subgroup $G\subset GL(2,\mathbb{C})$, by the Theorem above).

We use the same notation $\pi\colon\mathbb{C}^2\to W$ as before, π is birational, so we can embed $\mathbb{C}^2\subset\tilde{W}$ as a Zariski dense open subset of a quasi-projective, smooth surface $\tilde{W}$ having a proper morphism $\tilde{\pi}\colon\tilde{W}\to W$ $(\tilde{\pi}|_{\mathbb{C}^2}=\pi)$. Then $\tilde{W}$ is obtained by a succession of blowing-ups from the minimal resolution of singularity at the vertex of W. Since W has a rational singularity at the vertex, the minimal resolution of singularity has already a divisor with normal crossings as the exceptional divisor (where no three irreducible curves pass through a common point). Since $\pi^{-1}(p)=\{H=0\}$ where $H=\text{g.c.d.}\,(F_1,\dots,F_n)$, the Proposition is proved. The last assertion follows from the Theorem.

Remark 4. Combining Prop. 1 and the Theorem, we have an affirmative answer to (*) in case R is birational with $\mathbb{C}[X,Y]$.

The next result is somewhat striking.

PROPOSITION 2.

Let $\mathbb{C}\subset R\subset\mathbb{C}[X,Y]$ be a graded subring of dimension 2. Suppose that $\mathbb{C}[X,Y]$ is not

integral over R. Then the integral closure $\bar{R}$ of R in $\mathbb{C}[X, Y]$ is generated by homogeneous polynomials H, $HG_2, \ldots, HG_n$ where H has a positive degree. (Equivalently, any 2-dimensional graded subring of $\mathbb{C}[X, Y]$ of the form $\mathbb{C}[HG_1, \ldots, HG_n]$ with $H, G_1, \ldots, G_n$ all of positive degrees and g.c.d. $(G_1, \ldots, G_n) = 1$ is either non-normal or has quotient field strictly smaller than $\mathbb{C}(X, Y)$.)

Proof. Assume $\bar{R} = \mathbb{C}[HG_1, \ldots, HG_n]$ where $H, G_1, \ldots, G_n$ are non-constant homogeneous polynomials such that g.c.d. $(G_1, \ldots, G_n) = 1$. By Lemma 3, $\mathbb{C}[X, Y]$ is integral over $\mathbb{C}[G_1, \ldots, G_n]$. In particular, we have an integral equation satisfied by H: $H^N + a_1(G_1, \ldots, G_n)H^{N-1} + \cdots + a_N = 0$ with $a_i(T_1, \ldots, T_n)$ quasi-homogeneous polynomials in the indeterminates $T_1, \ldots, T_n$ such that $a_i(G_1, \ldots, G_n)$ are homogeneous polynomials in X, Y.

Let $\deg a_i(T_1, \ldots, T_n) = d_i$. Then $H^{d_i}a_i(G_1, \ldots, G_n)$ is a polynomial in $H, HG_1, \ldots, HG_n$ whose H-degree $\leqslant d_i - 1$. Let $d = \max d_i$. Now

$$H^{N+d} + H^{N-1+d}a_1(G_1, \ldots, G_n) + \cdots + H^d a_N(G_1, \ldots, G_n) = 0.$$

We have

$$\begin{aligned} H^{N-i+d}a_i(G_1, \ldots, G_n) &= H^{N-i+d-d_i}[H^{d_i} \cdot a_i(G_1, \ldots, G_n)] \\ &= H^{N-i+d-d_i}b_i(H, HG_1, \ldots, HG_n). \end{aligned}$$

$b_i(H, HG_1, \ldots, HG_n)$ is a polynomial in $H, HG_1, \ldots, HG_n$ with H-degree $\leqslant d_i - 1$. Thus $H^{N-i+d-d_i}b_i(H, HG_1, \ldots, HG_n)$ is a polynomial in H, $HG_1, \ldots, HG_n$ with H-degree $\leqslant N - i + d - d_i + d_i - 1 = N - i + d - 1$. For $i \geqslant 1$, $N + d > N - i + d - 1$. This shows therefore that H is integral over $\mathbb{C}[HG_1, \ldots, HG_n]$. Since this ring is assumed to be normal and has quotient field $\mathbb{C}(X, Y)$ $H \in \mathbb{C}[HG_1, \ldots, HG_n]$. This is clearly a contradiction.

3. Examples

1) Consider the $\mathbb{C}^*$ action on $\mathbb{C}[X, Y, Z, W]$ given by $t(X, Y, Z, W) = (tX, tY, t^{-1}Z, t^{-1}W)$ for $t \in \mathbb{C}^*$. The ring of invariants is $R = \mathbb{C}[XZ, XW, YZ, YW]$. This is isomorphic to $\mathbb{C}[U, V, S, T]/(UV - ST)$. This ring has an ordinary double point singularity at the origin, with trivial local fundamental group. Obviously the ring is not a UFD. In fact, one can show easily that the height 1 prime $(\bar{U}, \bar{S})$ has infinite order in the divisor class group. It follows that R cannot be isomorphic to a ring of invariants $\mathbb{C}[X_1, X_2, X_3]^W$ for any finite group, because any such ring of invariants has torsion divisor class group.

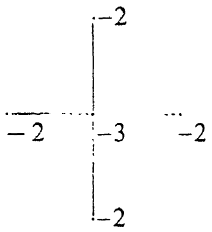
This example shows that Wall's conjecture does not have an analogue in dimension > 2.

2) One can prove the following statement using Pinkham's construction of a normal affine surface with a good $\mathbb{C}^*$ action:

Let V be a normal affine surface/$\mathbb{C}$ with a good $\mathbb{C}^*$ action such that the canonical equivariant resolution has the dual graph with $\mathbb{P}^1$ as the central curve. Then the co-ordinate ring, $\Gamma(V)$, of V is isomorphic to a graded subring of $\mathbb{C}[X, Y]$. For the definitions and general properties of surfaces with $\mathbb{C}^*$ actions, see [7].

We will illustrate the proof of above statement by considering the simplest rational

singularity with a good $\mathbb{C}^*$ which is not a quotient singularity. Consider the dual graph



where all the curves are isomorphic to $\mathbb{P}^1$. From [8], one sees that there is a normal surface V with a good $\mathbb{C}^*$ action with the vertex having a rational singularity whose minimal (as also the canonical equivariant) resolution has the above dual graph.

In [8], Pinkham has calculated the coordinate ring, $\Gamma(V)$, of V. Let t be an affine coordinate in $\mathbb{P}^1$-$\{\infty\}$. Then the graded ring $\Gamma(V)$ is generated by the following elements:

$$\left.\begin{aligned} x_1 &= t(t-1)(t-a) \\ x_2 &= t^2(t-1)(t-a) \\ x_3 &= t^3(t-1)(t-a) \end{aligned}\right\} \text{ degree 2 elements}$$

$$\left.\begin{aligned} y_1 &= t^2(t-1)^2(t-a)^2 \\ y_2 &= t^3(t-1)^2(t-a)^2 \end{aligned}\right\} \text{ degree 3 elements}$$

Here $a \in \mathbb{C} - \{0, 1\}$. Then $\Gamma(V)$ can be considered as the following subring of $k[X, t]$:

$$\Gamma(V) \approx \mathbb{C}[X^6 x_1, X^6 x_2, X^6 x_3, X^9 y_1, X^9 y_2].$$

Write $X \cdot t = Y$. Then all the five elements above are homogeneous polynomials in X and Y. For example

$$\begin{aligned} X^9 \cdot y_2 &= X^2 (Xt)^3 (Xt - X)^2 (Xt - aX)^2 \\ &= X^2 \cdot Y^3 (Y - X)^2 (Y - aX)^2, \text{ etc.} \end{aligned}$$

We thus get

$$\begin{aligned} \Gamma(V) \approx \mathbb{C}[&X^3 Y(Y-X)(Y-aX), X^2 Y^2 (Y-X)(Y-aX), \\ &X Y^3 (Y-X)(Y-aX), X^3 \cdot Y^2 (Y-X)^2 (Y-aX)^2, \\ &X^2 Y^3 (Y-X)^2 (Y-aX)^2]. \end{aligned}$$

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References

[1] Anderson D F, Subrings of $k[X, Y]$ generated by monomials, *Can. J. Math.* Vol. XXX, No. 1 (1978) 215–224

[2] Bourbaki N, Commutative Algebra. Hermann, Addison Wesley Publishers

[3] Gurjar R V and Shastri A R, The fundamental group at infinity of affine surfaces, *Commen. Math. Helvetici* **59** (1984) 459–484

[4] Gurjar R V and Shastri A R, A topological characterization of $\mathbb{C}^2/G$, *J. Math. Kyoto Univ.* **25**, No. 4 (1985) 767–773

[5] Huang H, Subrings of $k[X_1, \ldots, X_n]$ generated by monomials. Preprint

[6] Miyanishi M, Normal affine subalgebras of a polynomial ring, Volume dedicated to the memory of T. Miyata, (Kinokuniya-North Holland)

[7] Orlik P and Wagreich P, Isolated singularities of algebraic surfaces with $\mathbb{C}^*$-action, *Ann. Math.* **93** (1971) 205–226

[8] Pinkham H, Normal surface singularities with $\mathbb{C}^*$-action, *Math. Ann.* **227** (1977) 183–193

[9] Riemenschneider O, Die Invarianten der Endlichen Untergruppen von GL(2, $\mathbb{C}$), *Math. Z.* **153** (1977) 137–150 No. 1

[10] Wall C T C, Functions on quotient singularities, *Philos. Trans. R. Soc. London* **324** (1987) 1–45

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A note on the coefficient rings of polynomial rings

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Abstract. Let A and B be two reduced commutative rings with finitely many minimal prime ideals. If the polynomial algebras $A[X_1 \cdots X_n] = B[Y_1 \cdots Y_n]$ where X_i, Y_i are variables over A and B respectively, then there exists an injective ring homomorphism $\phi: A \to B$ such that B is finitely generated over $\phi(A)$.

Keywords. Polynomial ring; integral extension; minimal primes.

1. Introduction

In [1] the following question is posed. If $A[X_1 \cdots X_n] = B[Y_1 \cdots Y_n]$, do there exist isomorphisms of A into B and B into A? In particular, does there exist an isomorphism $K_i =$ quotient field of A/p_i. Then it is easy to see from the proof of the Noether's normalization lemma (1.1) in [2] that there exists positive integers $r_1, \ldots, r_n$ such that rings with finitely many minimal prime ideals.

2. Some lemmas

All rings are assumed to be commutative rings with identity. In this paper, we state the following version of Noether's normalization lemma (for proof see [2]).

Noether's normalization lemma 2.1. Let B be an affine integral domain, generated by $\{b_i\}_0^n$ over a field K. If $\mathrm{tr}(B:K) = 1$, then

$$K[b] \to B$$

is an integral extension where $b = b_0 + b_1^{r_1} + \cdots + b_n^{r_n}$ for some positive integers $r_1 \cdots r_n$.

Lemma 2.2. Let $A \to B$ be a ring extension of reduced rings with finitely many minimal prime ideals. Then following are equivalent.

(a) *A and B have equal number of minimal prime ideals.*
(b) *Any minimal prime ideal of B contracts to a minimal prime of A and distinct minimal primes of B contract to distinct minimal primes of A.*

Proof. Suppose the set of minimal primes of B is $\{q_1 \cdots q_m\}$. Since $\cap q_i = 0$, (b)$\Rightarrow$(a) is obvious. To prove (a)$\Rightarrow$(b) it is enough to show that non zero divisors of A remain

so in B. Let $c \in q_i \cap A$ for some i, $1 \leqslant i \leqslant m$. Now write $p_j = q_j \cap A$, $j = 1 \cdots i-1$, $i+1, \ldots, m$. Since A has m minimal primes

$$I = \bigcap_{\substack{j=1 \\ j \neq i}}^{m} p_j \neq 0.$$

Let $0 \neq c' \in I$. Then $cc' \in q_i \forall i$, and so $cc' = 0$ showing that only a zero divisor of A goes to a zero divisor of B.

Lemma 2.3. *Let $A \to B$ be a finitely generated ring extension of reduced rings with finitely many minimal prime ideals. Suppose A and B have an equal number of minimal prime ideals and that* $\mathrm{tr}\,(B/q{:}A/A \cap q) = 1$ *for any minimal prime q of B. If B is generated by $\{b_0, b_1 \cdots b_n\}$ over A, then*

$$A_a[b] \to B_a$$

is an integral extension where, $b = b_0 + b_1^{r_1} + \cdots + b_n^{r_n}$ for some integers $r_i > 0$ and a is some non zero divisor in A.

Proof. Suppose $\{q_1 \cdots q_m\}$ is the set of minimal prime ideals of B. Set $p_i = q_i \cap A$ and $k_i =$ quotient field of A/p_i. Then it is easy to see from the proof of the Noether's normalization lemma (1.1) in [M] that there exists positive integers $r_1, \ldots, r_n$ such that

$$K_i[b_i'] \to (B/q_i) \otimes K_i$$

is an integral extension where b_i' is a natural image of b in $(B/q_i) \otimes K_i$ and

$$b' \equiv b_0 + b_1^{r_1} + \cdots + b_n^{r_n} \pmod{q_j}.$$

Since B/q_i is finitely generated over A/p_i, we can find $a_i \in A$, $a_i \notin q_i$ such that

$$(A/p_i)_{\bar{a}_i}[b_i'] \to (B/q_i)_{\bar{a}_i},$$

is an integral extension.

Choose $a_i' \in \cap_{j \neq i} p_j$, $a_i' \neq 0$ and set $a = \sum a_i a_i'$, then

$$(A/p_i)_{\bar{a}}[b] \to (B/q_i)_{\bar{a}}$$

is an integral extension for $i = 1 \ldots m$.

Consider the following commutative diagram.

$$\begin{tikzcd}
A_a[b] \arrow[r] \arrow[d, "\eta_1"'] & B_a \arrow[d, "\eta_2"] \\
\bigoplus_{i=1}^{m} (A/p_i)_{\bar{a}}[b] \arrow[r, "\alpha'"] & \bigoplus_{i=1}^{m} (B/q_i)_{\bar{a}}
\end{tikzcd}$$

where η_1 and η_2 are natural injections and α is obtained by the product of the injections $(A/p_i)_{\bar{a}}[b] \to (B/q_i)_{\bar{a}}$. Note that $\oplus (A/p_i)[b]_{\bar{a}}$ is finitely generated as a module over $A_a[b]$ and $\oplus (B/q_i)_{\bar{a}}$ is integral over $(\oplus (A/p_i)[b]_{\bar{a}})$. Therefore $\oplus (B/q_i)_{\bar{a}}$ is integral over $\alpha(A_a[b])$. But $B_a \hookrightarrow \oplus (B/q_i)_a$. Hence $A_a[b] \to B_a$ is integral.

3. Main result

We will prove the following proposition first and then we will derive the main result as a corollary.

PROPOSITION 3.1.

Let A and B be two reduced rings with finitely many and an equal number of minimal prime ideals. Suppose $A \to B[Y]$ is a finitely generated extension with $\mathrm{tr}(B/q[Y]: A/A \cap q) = 1$ for all minimal primes q of B. Then there exists an injection $\phi: A \to B$ such that B is finitely generated over $\phi(A)$ and $\phi(A)_a \to B_a$ is integral extension for some nonzero divisor $a \in \phi(A)$.

Proof. Suppose $\{f_i\}$ is a finite set of generators of $B[Y]$ over A and suppose $\{b_i \cdots b_n\}$ is the set of coefficients of f_i in B for all i. Then $B[Y]$ is generated by $\{Y, b_i \cdots b_n\}$ over A. By lemma 2.3, there exists a nonzero divisor $a \in A$ such that

$$A_a[Y + b_1^{r_1} + \cdots + b_n^{r_n}] \to B[Y]_a$$

for some r_i $(i = 1 \cdots n)$ is integral. Since $B[Y] = B[Y + \Sigma b_i^{r_i}]$, we can replace $Y + b_1^{r_1} + \cdots + b_n^{r_n}$ by Y to get

$$A_a[Y] \to B[Y]_a \text{ integral.}$$

First note that Y is transcendental over A – i.e. Y has no algebraic relation over A. We now show that the composition ϕ

$$\underbrace{A \to A[Y] \to B[Y] \to}_{\phi} \frac{B[Y]}{YB[Y]}$$

is injective.

It is enough to show that $YB[Y] \cap A = 0$. Let $c \in YB[Y] \cap A$, be non zero. Then $c = Y \cdot f(Y)$ for some $f \in B[Y]$. Since f is integral over $A_a[Y]$, for a large power s there is a relation

$a^s f^t + g_i(Y) f^{t-1} + \cdots + g_t(Y) = 0$ with $g_i(Y) \in A[Y]$ for $i = 1 \cdots t$. Multiplying by Y^t, we get

$$a^s c^t + Y g_1(Y) c^{t-1} + \cdots + Y^t g_t(Y) = 0.$$

Since a is a non zero divisor and $c \neq 0$, this gives a relation for Y over A—which is absurd. This completes the proof.

Theorem 3.2. *Let A and B be two reduced rings with finitely many minimal prime ideals. If the polynomial rings*

$$A[X_1, \ldots, X_n] = B[Y_1, \ldots, Y_n]$$

then there exists an injection $\phi: A \to B$ such that B is finitely generated over $\phi(A)$.

Proof. Set $A_i = A[X_1 \cdots X_i]$ and $B_i = B[Y_1 \cdots Y_i]$. Then $B_{n-1}[Y_n]$ is finitely generated over A_{n-1} and for any minimal prime q of B_n,

$$\mathrm{tr}(B_n/q : A_{n-1}/q \cap A_{n-1}) = 1.$$

Since A and B must have an equal number of minimal prime ideals, by proposition (3.1)$\exists \phi_1 : A_{n-1} \to B_{n-1}$ such that B_{n-1} is finitely generated over ϕ_1 (A_{n-1}) and

$$\phi_1(A_{n-1})_a \to (B_{n-1})_a$$

is an integral extension where $a \in \phi_1(A_{n-1})$ is a nonzero divisor. Consider $\phi_1(A_{n-2}) \to B_{n-1} = B_{n-2}[Y_{n-1}]$. For any minimal prime q of B_{n-1},

$$\mathrm{tr}\left(\frac{B_{n-1}}{q} : \frac{\phi(A_{n-2})}{q \cap \phi(A_{n-2})}\right) = \mathrm{tr}\left(\frac{B_n}{q[Y_n]} : \frac{\phi(A_{n-1})}{q[Y_n] \cap \phi(A_{n-1})}\right) = 1.$$

Also, A_{n-2} and B_{n-1} both have equal number of minimal primes as they are polynomial rings over A and B respectively. Again by proposition (3.1), $\exists \phi_2 : \phi_1(A_{n-2}) \to (B_{n-2})$ such that B_{n-2} is finitely generated over $\phi_1(A_{n-2})$ and for some nonzero divisor $a \in \phi_1(A_{n-2})$ $\phi_2 \circ \phi_1(A_{n-2})_a \to (B_{n-2})_a$ is an integral extension. Using the same process repeatedly, we can complete the proof.

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References

[1] Abhyankar S S, Eakin P and Heinzer W, On the uniqueness of the coefficient ring in a polynomial ring, *J. Alg.* **23**/2 (1972) pp 310–342

[2] Mumford D, *Introduction to Algebraic Geometry* (Preliminary version of first three chapters) (Cambridge: Harvard University)

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Ramifications of Ramanujan's work on η-products

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Abstract. We follow, the evolution of ideas arising from Ramanujan's 1916 paper 'On certain arithmetical functions' by examining multiplicative η-products and quotients and their relation with the characters of the Mathieu group M_{24} and the automorphism group of the Leech lattice. This leads to the Monster and speculations on its geometric origin and current physics.

Keywords. Finite simple groups; products of η-functions; Mathieu group M_{24}; Leech lattice.

1. Introduction

In 1916 Ramanujan published his paper [18] "On certain arithmetical functions" in which he investigated the properties of several modular forms with multiplicative q-coefficients amongst which is the famous function

$$\Delta(z) = q \prod_{i \geq 1} (1 - q^i)^{24} = \eta(z)^{24},$$

where $\eta(z)$ is the Dedekind η-function and $q = \exp(2\pi i z)$. We recount here the remarkable connections with number theory and finite group theory and ultimately quantum field theory which can be viewed as closely related to his work.

2. Finite simple groups

To set the scene we recall that the decade starting at the mid-1960's saw the discovery of all twenty-one modern sporadic finite simple groups, from the smallest – Janko's group J_1, $|J_1| = 175560 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$, to the largest – an enormous group commonly called the Friendly Giant or Monster F_1,

$$\begin{aligned}|F_1| &= 808017424794512875886459904961710757005754368000000000\\ &= 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71.\end{aligned}$$

Apart from these modern groups, Mathieu [14] in 1861, found the five permutation groups M_{11}, M_{12}, M_{22}, M_{23}, and M_{24} of degree given by the subscripts. Burnside [1] in his Note N, first recorded the name "sporadic", where he says of the Mathieu groups – "These apparently sporadic groups would probably repay a closer examination

than they have yet received." – How prophetic! Evidence that the largest Mathi group astonished contemporary mathematicians is found in the paper [15] G A Miller appearing in 1898 (unfortunately not contained in his collected worl in which he purports to prove that the group does not exist.

We now know that the finite simple groups consist of the cyclic and alternati groups, groups of Lie type and their fixed-point groups, and the twenty-six sporad groups.

3. Products of η-functions

For $f(z)=\prod_{t\geqslant 1}\eta(tz)^{e_t}$ we write $(1^{e_1},2^{e_2},\ldots)$ with $\mathrm{Im}(z)>0$ and with finitely $e_t \neq$ Expanding $f(z)=\sum_{n\geqslant 1}a_n q^n$ as a Fourier series, we say $f(z)$ is multiplicative if $a_1 \neq$ and $a_1 a_{mn}=a_m a_n$ when m and n are relatively prime. We call $f(z)$ a product if all e_t a non-negative; if not, it is called a quotient. Which η-products are multiplicative?

By examining the initial coefficients of the η-products coded by the 1575 partitio of 24 we find thirty possible candidates and are then able to prove the multiplicati property and identify the corresponding Dirichlet series as L-series, some w Grössen-character (see [4]).

We define the shape of a permutation to be the partition of its degree formed its disjoint cycle lengths. This partition is also the code for an η-product and so have

$$x\mapsto \mathrm{shape}(x)=(1^{e_1},2^{e_2},\ldots,r^{e_r})\mapsto f_x(z)=\prod_{t\geqslant 1}\eta(tz)^{e_t}.$$

If the product is multiplicative then the Mellin transform of $f_x(z)$ is the Euler produ

$$\prod_p(1-a_p(x)p^{-s}+b_p(x)p^{-2s})^{-1}$$

with $b_p(x)=\varepsilon_p(x)p^{k-1}$, $\varepsilon_p(x)=(-N/p)^k$, where N is the product of the largest a smallest cycles in shape(x), $(-N/p)$ is a quadratic character, and $k=k(x)$ $(=r/2$ wh r is even) is the weight of $f_x(z)$ as a modular form.

4. The Ramanujan characters of M_{24}

Remarkably, all twenty-one shapes of elements $x\in M_{24}$ have r even and f_x multiplicative. Since $f_x(z)=\prod_{t\geqslant 1}P(q^t)$, where $P(q)$ is the characteristic polynom of x, it follows that $a_p(x)$ and $b_p(x)$ are values of virtual characters (that is, $\mathbb{Z}$-line combinations of irreducible characters) of M_{24}. This might be useful in trying prove Lehmer's conjecture that all the coefficients of $\Delta(z)$ $(=f_1(z))$ are non-zero. F all primes $p\neq 3$, b_p $(=a_p^2-a_{p^2})$ is a proper character of degree p^{11}.

By tensoring, if necessary, we have the remarkable slight generalization of a theore of Mason:

Theorem. *For every integer n there is a virtual representation of M_{24} of degree n with character values either zero or (to within sign) powers of n.*

This has been investigated by Mason [11], [12], [13], and this type of proper

Table 1. The rational character table of M_{24}.

		8	6	4		3	3		6 6	11	15	15	14	14								10			4	12
		2	3	5	4444	7	7	8 8	3 3	11	5	5	7	7								10			4	6
	24	8	6	4	22	3	3	4 2	2 2	1	3	3	2	2	23	23	2	4	6	8	12	2	21	21	4	4
Shape:	1	1	1	1	1111	1	1	1 1	1 1	1	1	1	1	1	1	1	12	6	4	3	2	2	3	3	2	2
Centralizer:	244823040	21504	1080	60	128	42	42	16	24	11	15	15	14	14	23	23	12	24	96	504	7680	20	21	21	384	12
Weight:	12	8	6	4	5	3	3	3	4	2	2	2	2	2	1	1	1	2	3	4	6	2	1	1	4	2
N:	1	2	3	5	4	7	7	8	6	11	15	15	14	14	23	23	144	36	16	9	4	20	63	63	8	24
1:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2:	23	7	5	3	3	2	2	1	1	1	.	.	.	.	.	.	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
* 3:	90	-6	.	.	2	-1	-1	-2	.	2	.	.	1	1	-2	-2	2	-2	2	6	10	.	-1	-1	-6	.
* 4:	462	14	-6	2	-2	.	.	-2	2	.	-1	-1	.	.	2	2	.	.	6	.	-18	2	.	.	-2	-2
5:	252	28	9	2	4	.	.	.	1	-1	-1	-1	.	.	-1	-1	.	.	.	.	12	2	.	.	4	1
6:	253	13	10	3	1	1	1	-1	-2	.	.	.	-1	-1	.	.	1	1	1	1	-11	-1	1	1	-3	.
7:	483	35	6	-2	3	.	.	-1	2	-1	1	1	.	.	.	.	.	.	3	.	3	-2	.	.	3	.
* 8:	1540	-28	10	.	-4	.	.	.	2	.	.	.	.	.	-1	-1	2	2	-4	-14	20	.	.	.	4	-2
* 9:	1980	-36	.	.	4	-1	-1	.	.	.	.	.	-1	-1	2	2	2	-2	-4	6	-20	.	-1	-1	12	.
*10:	2070	-42	.	.	6	-2	-2	-2	.	2	.	.	.	.	.	.	-2	2	-2	-6	-10	.	1	1	6	.
11:	1035	27	.	.	-1	-1	-1	1	.	1	.	.	-1	-1	.	.	.	2	3	6	35	.	-1	-1	3	.
12:	1265	49	5	.	1	-2	-2	1	1	.	.	.	.	.	.	.	.	.	-3	8	-15	.	1	1	-7	-1
13:	1771	-21	16	1	-5	.	.	-1	.	.	1	1	.	.	.	.	-1	-1	-1	7	11	1	.	.	3	.
14:	2024	8	-1	-1	.	1	1	.	-1	.	-1	-1	1	1	.	.	.	.	.	8	24	-1	1	1	8	-1
15:	2277	21	.	-3	1	2	2	-1	.	.	.	.	.	.	.	.	.	2	-3	6	-19	1	-1	-1	-3	.
16:	3312	48	.	-3	.	1	1	.	.	1	.	.	-1	-1	.	.	.	-2	.	-6	16	1	1	1	.	.
17:	3520	64	10	.	.	-1	-1	.	-2	.	.	.	1	1	1	1	.	.	.	-8	.	.	-1	-1	.	.
18:	5313	49	-15	3	-3	.	.	-1	1	.	.	.	.	.	.	.	.	.	-3	.	9	-1	.	.	1	1
19:	5544	-56	9	-1	.	.	.	.	1	.	-1	-1	.	.	1	1	.	.	.	.	24	-1	.	.	-8	1
20:	5796	-28	-9	1	4	.	.	.	-1	-1	1	1	.	.	.	.	.	.	.	.	36	1	.	.	-4	-1
21:	10395	-21	.	.	-1	.	.	1	.	.	.	.	.	.	-1	-1	.	.	3	.	-45	.	.	.	3	.

* denotes the sum of two complex irreducible characters

Table 2. The η-product q-coefficients.

	24 1	8 2 8 1	6 3 6 1	4 5 4 1	4444 22 1111	3 7 3 1	3 7 3 1	8 8 4 2 1 1	6 6 3 3 2 2 1 1	11 11 1 1	15 5 3 1	15 5 3 1	14 7 2 1	14 7 2 1	23 1	23 1	2 12	4 6	6 4	8 3	12 2	10 10 2 2	21 3	21 3	4 4 4 2	12 6 4 2
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	-24	-8	-6	-4	-4	-3	-3	-2	-2	-2	-1	-1	-1	-1	-1	-1	·	·	·	·	·	·	·	·	·	·
3	252	12	9	2	·	·	·	-2	-3	-1	-1	-1	-2	-2	-1	-1	·	·	·	·	-12	-2	·	·	-4	-1
4	-1472	64	4	8	16	5	5	4	4	2	-1	-1	1	1	·	·	·	·	·	-8	·	·	-1	-1	·	·
5	4830	-210	6	-5	-14	·	·	·	6	1	1	1	·	·	·	·	·	·	-6	·	54	-1	·	·	-2	-2
6	-6048	-96	-54	-8	·	·	·	4	6	2	1	1	2	2	1	1	·	·	·	·	·	·	·	·	·	·
7	-16744	1016	-40	6	·	-7	-7	·	-16	-2	·	·	1	1	·	·	·	-4	·	20	-88	2	-1	-1	24	·
8	84480	-512	168	·	-64	-3	-3	-8	-8	·	3	3	-1	-1	1	1	·	·	·	·	·	·	·	·	·	·
9	-113643	-2043	81	-23	81	9	9	-5	9	-2	1	1	1	1	·	·	·	·	9	·	-99	1	·	·	-11	1
10	-115920	1680	-36	20	56	·	·	·	-12	-2	-1	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
11	534612	1092	-564	32	·	-6	-6	14	12	1	-4	-4	·	·	·	·	·	·	·	·	540	·	·	·	-44	4
12	-370944	768	36	16	·	·	·	-8	-12	-2	1	1	-2	-2	·	·	·	·	·	·	·	·	·	·	·	·
13	-577738	1382	638	-38	-238	·	·	·	38	4	-2	-2	-4	-4	-1	-1	-2	2	10	-70	-418	2	·	·	22	-2
14	401856	-8128	240	-24	·	21	21	·	32	4	·	·	-1	-1	·	·	·	·	·	·	·	·	·	·	·	·
15	1217160	-2520	54	-10	·	·	·	·	-18	-1	-1	-1	·	·	·	·	·	·	·	·	-648	2	·	·	8	2
16	987136	4096	-1136	-64	256	-11	-11	16	16	-4	-1	-1	1	1	-1	-1	·	·	·	64	·	·	1	1	·	·
17	-6905934	14706	882	26	322	·	·	2	-126	-2	2	2	6	6	·	·	·	·	-30	·	594	-6	·	·	50	2
18	2727432	16344	-486	92	-324	-27	-27	10	-18	4	-1	-1	-1	-1	·	·	·	·	·	·	·	·	·	·	·	·
19	10661420	-39940	-556	100	·	·	·	-34	20	·	4	4	2	2	·	·	·	8	·	56	836	-4	·	·	44	-4
20	-7109760	-13440	24	-40	-224	·	·	·	24	2	-1	-1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
21	-4219488	12192	-360	12	·	·	·	·	48	2	·	·	-2	-2	·	·	·	·	·	·	1056	-4	·	·	-96	·
22	-12830688	-8736	3384	-128	·	18	18	-28	-24	-2	4	4	·	·	·	·	·	·	·	·	·	·	·	·	·	·
23	18643272	68712	-840	-78	·	18	18	·	168	-1	·	·	·	·	1	1	·	·	·	·	-4104	6	·	·	-56	-8
24	21288960	-6144	1512	·	·	·	·	16	24	·	-3	-3	2	2	-1	-1	·	·	·	·	·	·	·	·	·	·
25	-25499225	-34025	-3089	25	-429	25	25	25	-89	-4	1	1	-5	-5	1	1	-1	-5	11	-125	-209	1	1	1	-121	-1
26	13865712	-11056	-3828	152	952	·	·	·	-76	-8	2	2	4	4	1	1	·	·	·	·	·	·	·	·	·	·
27	-73279080	-50760	729	-100	·	·	·	28	-27	5	-1	-1	4	4	1	1	·	·	·	·	4104	4	·	·	152	-1
28	24647168	65024	-160	48	·	-35	-35	·	-64	-4	·	·	1	1	·	·	·	·	·	-160	·	·	1	1	·	·
29	128406630	-102570	4638	-50	82	-54	-54	·	30	·	-2	-2	-6	-6	-1	-1	·	·	42	·	-594	6	·	·	198	6
30	-29211840	20160	-324	40	·	·	·	·	36	2	1	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·
31	-52843168	227552	4400	-108	·	·	·	·	-88	7	·	·	-4	-4	-1	-1	·	-4	·	308	4256	-4	·	·	-160	8
32	-196706304	-32768	1440	256	-1024	45	45	-32	-32	8	-5	-5	-1	-1	·	·	·	·	·	·	·	·	·	·	·	·

Table 2. *(continued)*

	24 1	8 2 8 1	6 3 6 1	4 5 4 1	4444 22 1111	3 7 3 1	3 7 3 1	8 8 4 2 1 1	6 6 3 3 2 2 1 1	11 11 1 1	15 5 3 1	15 5 3 1	14 7 2 1	14 7 2 1	23 1	23 1	2 12	4 6	6 4	8 3	12 2	10 10 2 2	21 3	21 3	4 4 4 2	12 6 4 2
33	134722224	13104	−5076	64	.	.	.	−28	−36	−1	4	4	.	.	.	.	.	.	.	.	−6480	.	.	.	176	−4
34	165742416	−117648	−5292	−104	−1288	.	.	−4	252	4	−2	−2	−6	−6	.	.	.	.	.	.	.	.	.	.	.	.
35	−80873520	−213360	−240	−30	.	.	.	.	−96	−2	.	.	.	.	.	.	.	.	.	.	−4752	−2	.	.	−48	.
36	167282496	−130752	324	−184	1296	45	45	−20	36	−4	−1	−1	1	1	.	.	.	.	.	.	.	.	.	.	.	.
37	−182213314	160526	−2410	266	2162	−38	−38	.	254	3	−10	−10	2	2	.	.	2	−10	−70	110	−298	2	−2	−2	−162	6
38	−255874080	319520	3336	−400	.	.	.	68	−40	.	−4	−4	−2	−2	.	.	.	.	.	.	.	.	.	.	.	.
39	−145589976	16584	5742	−76	.	.	.	.	−114	−4	2	2	8	8	1	1	.	.	.	.	5016	−4	.	.	−88	2
40	408038400	107520	1008	.	896	.	.	.	−48	.	3	3	.	.	.	.	.	.	.	.	.	.	.	.	.	.
41	308120442	10842	−6870	22	−3038	.	.	−46	42	−8	10	10	6	6	−1	−1	.	.	18	.	17226	6	.	.	−198	−6
42	101267712	−97536	2160	−48	.	.	.	.	−96	−4	.	.	2	2	.	.	.	.	.	.	.	.	.	.	.	.
43	−17125708	−630748	9644	442	.	58	58	14	−52	−6	4	4	8	8	.	.	.	8	.	−520	−12100	−10	−2	−2	52	4
44	−786948864	69888	−2256	256	.	−30	−30	56	48	2	4	4	.	.	.	.	.	.	.	.	.	.	.	.	.	.
45	−548895690	429030	486	115	−1134	.	.	.	54	−2	1	1	.	.	.	.	.	.	−54	.	−5346	−1	.	.	22	−2
46	−447438528	−549696	5040	312	.	−54	−54	.	−336	2	.	.	.	.	−1	−1	.	.	.	.	.	.	.	.	.	.
47	2687348496	472656	−18672	−514	.	.	.	.	−96	8	8	8	−12	−12	−1	−1	.	.	.	.	−1296	−6	.	.	528	.
48	248758272	49152	−10224	−128	.	.	.	−32	−48	4	1	1	−2	−2	1	1	.	.	.	.	.	.	.	.	.	.
49	−1696965207	208713	−15207	−307	2401	49	49	49	−87	−3	−7	−7	1	1	1	1	1	9	49	57	−9063	−3	1	1	233	−7
50	611981400	272200	18534	−100	1716	−75	−75	−50	178	8	−1	−1	5	5	−1	−1	.	.	.	.	.	.	.	.	.	.
51	−1740295368	176472	7938	52	.	.	.	−4	378	2	−2	−2	−12	−12	.	.	.	.	.	.	−7128	12	.	.	−200	−2
52	850430336	88448	2552	−304	−3808	.	.	.	152	8	2	2	−4	−4	.	.	.	.	.	560	.	.	.	.	.	.
53	−1596055698	−1494018	33750	2	2482	−6	−6	.	198	−6	−10	−10	6	6	.	.	.	.	90	.	19494	−6	.	.	−242	−2
54	1758697920	406080	−4374	400	.	.	.	−56	54	−10	1	1	−4	−4	−1	−1	.	.	.	.	.	.	.	.	.	.
55	2582175960	−229320	−3384	−160	.	.	.	.	72	1	−4	−4	.	.	.	.	.	.	.	.	29160	.	.	.	88	−8
56	−1414533120	−520192	−6720	.	.	21	21	.	128	.	.	.	−1	−1	.	.	.	.	.	.	.	.	.	.	.	.
57	2686677840	−479280	−5004	200	.	.	.	68	−60	.	−4	−4	−4	−4	.	.	.	.	.	.	−10032	−8	.	.	−176	−4
58	−3081759120	820560	−27828	200	−328	162	162	.	−60	.	2	2	6	6	1	1	.	.	.	.	.	.	.	.	.	.
59	−5189203740	2640660	−18084	500	.	.	.	−82	−660	5	−4	−4	−6	−6	2	2	.	.	.	.	−7668	12	.	.	−668	4
60	−1791659520	−161280	216	−80	.	.	.	.	−72	−2	1	1	.	.	.	.	.	.	.	.	.	.	.	.	.	.
61	6956478662	827702	39758	−518	−6958	.	.	.	−538	12	−2	−2	8	8	.	.	2	14	−22	182	−34738	2	.	.	550	−2
62	1268236032	−1820416	−26400	432	.	.	.	.	176	−14	.	.	4	4	1	1	.	.	.	.	.	.	.	.	.	.
63	1902838392	−2075688	−3240	−138	.	−63	−63	.	−144	4	.	.	1	1	.	.	.	.	.	.	8712	2	.	.	−264	.
64	2699296768	262144	27712	−512	4096	−91	−91	64	64	−8	7	7	1	1	1	1	.	.	.	−512	.	.	−1	−1	.	.
65	−2790474540	−290220	3828	190	3332	.	.	.	228	4	−2	−2	.	.	.	.	.	.	−60	.	−22572	−2	.	.	−44	4

(continued)

Table 2. *(continued)*

24 1	8 2 8 1	6 3 6 1	4 5 4 1	4444 22 1111	3 7 3 1	3 7 3 1	8 8 4 2 1 1	6 6 3 3 2 2 1 1	11 11 1 1	15 5 3 1	15 5 3 1	14 7 2 1	14 7 2 1	23 1	23 1	2 12	4 6	6 4	8 3	12 2	10 10 2 2	21 3	21 3	4 4 4 2	12 6 4 2
−3233333376	−104832	30456	−256	.	.	.	56	72	2	−4	−4	.	.	.	.	.	.	.	.	.	.	.	.	.	.
−15481826884	−126004	−23068	126	.	−118	−118	62	884	−7	12	12	−4	−4	.	.	.	−16	.	−880	21812	2	2	2	188	−4
10165534848	941184	3528	208	5152	.	.	8	−504	−4	−2	−2	6	6	.	.	.	.	.	.	.	.	.	.	.	.
4698104544	824544	−7560	−156	.	.	.	.	−504	1	.	.	.	.	−1	−1	.	.	.	.	49248	−12	.	.	224	8
1940964480	1706880	1440	120	.	.	.	.	192	4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
9791485272	−1414728	−4248	412	.	114	114	.	792	−3	−8	−8	.	.	−1	−1	.	.	.	.	−46872	−12	.	.	728	8
−9600560640	1046016	13608	.	−5184	−27	−27	40	−72	.	3	3	−1	−1	.	.	.	.	.	.	.	.	.	.	.	.

Table 3: Decomposition of the η-product characters.

	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}	a_{18}	a_{19}	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}	a_{25}	a_{26}	a_{27}	a_{28}	a_{29}	a_{30}
1	1	−1	−1	1	.	1	−1	−1	1	.	1	−1	−1	2	−1	1	.	−2	−1	−2	2	−1	8	3	−10	3	−1	−1	−1	3
2	.	−1	.	2	.	.	−1	−2	1	2	.	.	−2	1	−2	−1	12	−2	−11	−11	5	4	26	9	−36	6	−25	19	−6	6
* 3	.	.	.	.	.	.	.	.	1	.	.	.	−3	1	.	1	−1	−4	10	−1	−1	.	−9	4	−6	11	−6	−5	38	−8
* 4	.	.	.	.	.	.	.	.	−1	.	2	.	2	−2	.	2	−14	12	2	−9	2	−26	54	14	−27	12	−97	43	76	−18
5	.	.	.	1	−1	−1	1	−2	.	4	.	1	−1	−8	−3	4	26	12	−41	−32	11	−2	99	28	−114	.	−132	108	42	−4
6	.	.	1	.	−1	−2	1	2	−1	3	−5	3	.	−5	2	−11	22	10	−14	−21	−4	11	42	29	−69	−7	−112	69	107	−22
7	.	.	.	.	.	.	.	−1	1	2	−2	.	3	−8	−3	11	12	16	−45	−38	13	−16	158	41	−143	.	−222	147	120	−25
* 8	.	.	.	.	1	.	−2	2	.	−3	−1	−2	6	9	4	−9	−36	.	59	−9	−20	−20	12	79	−69	15	−189	34	491	−105
* 9	.	.	.	.	.	.	.	.	2	−1	.	−2	−7	8	8	5	−35	−7	76	−21	−30	−45	22	91	−84	80	−257	45	611	−135
*10	.	.	.	.	.	.	−1	.	3	−1	.	−2	−8	9	8	5	−36	−11	86	−22	−31	−45	13	95	−88	91	−263	43	649	−143
11	.	.	.	.	.	.	1	.	−5	1	7	−1	.	−9	−1	9	−12	37	−2	−45	2	−68	145	83	−150	36	−352	185	417	−98
12	.	.	.	.	.	.	2	−1	−5	3	4	−1	2	−16	1	15	−2	49	−41	−68	6	−75	264	105	−231	33	−501	273	454	−105
13	.	.	.	−1	1	−1	−1	7	−1	−5	−6	−2	11	14	11	−25	−62	10	114	−30	−47	−34	48	181	−184	20	−466	111	1094	−235
14	.	.	.	.	.	.	1	.	−2		5	−3	−8	.	8	12	−44	27	76	−64	−31	−109	162	172	−222	115	−606	226	1017	−236
15	.	.	.	.	.	.	.	.	.	.	2	−3	−4	−1	11	17	−50	31	56	−79	−30	−118	257	191	−275	118	−739	285	1091	−254
16	.	.	.	.	.	.	1	.	−5	1	9	−4	−2	−10	10	24	−62	68	50	−124	−28	−186	402	274	−419	154	−1091	476	1508	−352
17	.	.	.	.	−1	−1	4	2	−8	4	5	−2	−1	−20	12	14	−36	86	27	−144	−32	−178	450	300	−489	138	−1195	556	1583	−366
18	.	.	.	.	.	.	2	.	−11	1	24	−6	−14	−13	18	37	−138	114	157	−181	−55	−351	570	427	−583	311	−1719	685	2484	−580
19	.	.	.	.	1	.	−5	4	4	−8	8	−11	−11	34	32	4	−192	13	341	−124	−123	−239	232	512	−517	305	−1511	384	3210	−716
[illegible]							1		1	2	18	−9	−33	16	29	34	−172	39	314	−160	−109	−321	333	498	−542	411	−1658	504	3130	−716

Table 4. Decomposition of the Ramanujan character of M_{24} of degree n^{11}.

n	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	1	4	17	4	66	80	436	499	2228	3058	10982	17093	42986	72920	161546
2	1	·	10	59	42	500	990	4698	9847	33663	69876	191293	383061	866940	1657782	3362021
* 3	·	−1	1	8	68	243	1581	5180	18390	49700	136574	321135	744353	1566124	3233573	6243704
* 4	·	1	2	61	346	2106	8022	31054	94076	275086	701086	1709846	3819472	8216501	16596062	32464435
5	·	2	14	209	380	3228	9130	38858	103764	320300	764846	1931155	4172848	9125434	18116770	35827948
6	1	·	18	116	384	2643	9210	35316	104263	307711	767904	1892997	4189941	9040455	18189690	35676167
7	·	·	12	270	708	5306	17112	69849	197840	595276	1465788	3640819	7992096	17341935	34712496	68291975
* 8	·	1	17	115	1141	5832	27161	95560	314962	885885	2336857	5597548	12738856	27095162	55334489	107529654
* 9	·	−2	17	140	1468	7255	34737	122926	404382	1137002	3004534	7194511	16375393	34832591	71138193	138242623
*10	·	−3	17	146	1536	7493	36309	128106	422754	1186680	3141108	7515608	17119697	36398715	74371693	144486241
11	·	6	18	317	1536	9492	36318	138897	422772	1231507	3141108	7656737	17119746	36785595	74371766	145407095
12	·	6	26	448	1878	12131	44538	172962	517188	1518977	3839136	9404372	20926626	45080625	90903674	178022495
13	1	3	48	307	2628	13884	62640	222174	724875	2048091	5374800	12907857	29301783	62406825	127273952	247535171
14	·	3	34	438	2994	16681	70986	261405	826638	2366307	6142560	14840244	33478046	71575245	145437082	283488132
15	·	·	38	527	3360	19124	79878	296397	930062	2672034	6910320	16728631	37663262	80612475	163617670	319146619
16	·	6	54	838	4896	28602	116178	435294	1352798	3903491	10051428	24385286	54782910	117398070	237989290	464553534
17	·	10	68	953	5218	31002	123796	465421	1438688	4161487	10682782	25957183	58228498	124872371	252945364	493986054
18	·	12	80	1190	7884	44667	185984	691857	2168750	6234481	16124400	39030173	87873856	188103150	381760512	744663189
19	·	−2	102	941	8212	42825	194782	698323	2265326	6411964	16825342	40424197	91706544	195430451	398382446	775032306
20	·	·	92	1055	8592	45591	203096	737181	2366590	6728073	17590188	42345797	95865952	204555885	416473008	810824146
21	·	5	178	1974	15404	83070	364722	1327361	4245924	12094271	31547570	76030102	171941456	367091149	746950834	1454749013

goes by the rubric of "multiplicative moonshine". We note that the powers of n given by one less than half the number of fixed points of x as a permutation degree 24. The decomposition of these characters for arbitrary n is unknown.

We give rational character table for M_{24}, tabulate the decomposition of characters of degree n^{11} for n up to 17 and the first 72 terms of $A_{n(x)}$ and decomposition for n up to 30.

5. Other groups

The automorphism group $\cdot O$ ("dotto") of the Leech lattice, which is the unique e unimodular integral lattice of rank 24 containing no vectors of squared lengt (see [3] for details), has M_{24} as a monomial subgroup and so $M_{24} \subset \cdot O \subset SL_{24}$ The characteristic polynomial of an element $x \in \cdot O$ is given by $\prod_{t \geqslant 1}(1-\lambda^t)^{e_t}$, e_t and finitely many e_t non-zero. We write that the Frame shape of x is $(1^{e_1}, 2^{e_2}, \ldots,$ and we may construct the η-quotient $f_x(z)$. A complete description of the multiplica functions of this type is not yet known but many have been found by Kondo [8

The next step is to move to F_1 which contains an involution for which the central has the form of an extra-special 2-group (see Gorenstein [5] for definitions) of order acted on by $\cdot 1 \simeq \cdot O/\langle \pm 1 \rangle$. In F_1 the character-generating functions are mod functions rather than modular forms (see [17]) and the appropriate function on identity is the Klein modular function $j_1(z) = j(z) = (1 + 240\sum_{n \geqslant 1}\sigma_3(n)q^n)^3/\Delta(z)$, wh the numerator is the cube of the normalized function $g_2(z)$ which is the theta funct of the sole even unimodular integral lattice of rank 8, namely the root lattice of t E_8. This suggested checking the values of coefficients of the cube root of j aga the dimensions of irreducible representations of E_8 and finding that the in coefficients correspond to small linear combinations of dimensions of irreduc representations of $E_8(\mathbb{C})$. This was almost immediately afterwards proved for coefficients by Kac [7].

The details of Moonshine for F_1 were worked out by Conway-Norton [2] Thompson [20] who found that the Hauptmoduln $j_m(z)$, $m \in F_1$, generating characters of F_1 on the 172 rational conjugacy classes, are of genus zero (see [16]). This astonishing observation represents a challenge to our understandin F_1. Using Mahler's work [9], [10], we can prove that if a finite initial sequenc q-coefficients yields virtual characters of F_1 then all do (see also Smith [19]).

6. Future directions

It appears likely to us that the moonshine phenomena may be explicable within framework of techniques developed by Witten [21] and others in establis conformal quantum field theory. This involves a generalization of a trace-val version of the Atiyah-Singer index theorem to infinite-dimensional spaces. character-generating functions for G, a group acting on a manifold, are mod invariants as a consequence of properties of Feynman path integrals. We have mo from the algebra of a group to the geometry of a manifold. Hirzebruch [6] exhib a family of manifolds M_0^{4k} (due to Kervaire-Milnor). The smallest dimensi

manifold M_0^8 has elliptic genus given by the cube root of j. The possibility that the symmetries of some manifold cobordant to M_0^{24} may yield F_1, with elliptic genus the dimension-generating function $j(z)$, is tantalizing.

References

[1] Burnside W, *The theory of groups* 2nd ed. Note N (1911)

[2] Conway J H and Norton S P, Monstrous moonshine, *Bull. London Math. Soc.* (1979) 308–339

[3] Conway J H and Sloane N J A, *Sphere-packings, lattices and groups* (Springer-Verlag) (1988)

[4] Dummit D, Kisilevsky H and McKay J, Multiplicative products of η-functions, *Contemp. Math.* **45** 89–98

[5] Gorenstein D, *Finite groups* (Harper & Row) (1968)

[6] Hirzebruch F, Some remarks on elliptic genera of almost parallelizable manifolds, *Talk at Schloss Ringberg* (March 1987)

[7] Kac V G, An elucidation of: "Infinite-dimensional algebras, Dedekind's η-function, classical Möbius function and the very strange function formula." $E_8^{(1)}$ and the cube root of the modular invariant j, *Adv. Math.* **35** (1980) 264–273

[8] Kondo T, Examples of multiplicative η-products, *Fac. Sci. U. Tokyo* (1986) 133–149

[9] Mahler K, On a class of non-linear functional equations connected with modular functions, *J. Aust. Math. Soc.* **22A** (1976) 65–120

[10] Mahler K, On a special non-linear functional equation, *Proc. R. Soc. London* **A378** (1981)

[11] Mason G, Frame-shapes and rational characters of finite groups, *J. Alg.* **89** (1984) 237–246

[12] Mason G, M_{24} and certain automorphic forms, *Contemp. Math.* **45** (1985) 223–244

[13] Mason G, Finite groups and modular functions, *Proc. Symp. Pure Math.* (*Arcata*) **47** (1987) 181–210

[14] Mathieu E, Mémoire sur l'étude des fonctions de plusieurs quantités, *J. de Math.* **6** (1861) 242–323

[15] Miller G A, On the supposed five-fold transitive function of 24 elements and 19!/48 values, *Messenger Math.* **27** 188–190

[16] Ogg A P, Modular functions, *Proc. Symp. Pure Math.* (*Santa Cruz*) **37** (1980) 521–532

[17] Queen L, Modular functions and finite simple groups, *Proc. Symp. Pure Math.* (*Santa Cruz*) **37** (1980) 561–566

[18] Ramanujan S, On certain arithmetical functions, *Trans. Camb. Philos. Soc.* **22** (1916) 159–184

[19] Smith S, On the Head characters of the Monster simple group, *Contemp. Math.* **45** 303–313

[20] Thompson J G, Finite groups and modular functions, *Bull. London Math. Soc.* **11** (1979) 347–351

[21] Witten E, The index of the Dirac operator in loop space, *Proc. Conf. on elliptic curves and modular forms in Alg. Topology* Lecture notes in mathematics, 1326 (Springer) (1988).

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Transcendence conjectures about periods of modular forms and rational structures on spaces of modular forms

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Abstract. The conjecture is made that the rational structures on spaces of modular forms coming from the rationality of Fourier coefficients and the rationality of periods are not compatible. A consequence would be that $\zeta(2k-1)/\pi^{2k-1}$ ($\zeta(s)$ = Riemann zeta function; $k\in\mathbb{N}$, $k\geqslant 2$) is irrational or even transcendental.

Keywords. Modular forms; rational structures; periods; transcendence; Riemann zeta function.

The purpose of this note is to point out that some more or less standard and well-believed irrationality or transcendence conjectures about periods of modular forms of arbitrary weight are in fact simple consequences of a more general conjecture, which would say that for a number field K, the K-rational structures on spaces of modular forms given on the one hand by the rationality of periods and on the other hand by the rationality of Fourier coefficients are not compatible. In particular, when applied to the normalized Eisenstein series of weight $2k$ on $SL_2(\mathbb{Z})$ this conjecture would imply that the number $\zeta(2k-1)/\pi^{2k-1}$ ($\zeta(s)$ = Riemann zeta function; $k\in\mathbb{N}$, $k\geqslant 2$) is irrational or even transcendental.

Although our observation becomes rather obvious once the necessary definitions have been set up and essentially is an interpretation of some of the formulas already given in [3], §4·1, we feel that it is worthwhile being stated, especially since recently Beukers [2] found a proof of Apéry's result that $\zeta(3)$ is irrational by using period polynomials of Eisenstein series on congruence subgroups of $SL_2(\mathbb{Z})$.

Let us be more precise now. For $k\in\mathbb{N}$, $k\geqslant 2$ we denote by M_{2k} the complex vector space of modular forms of weight $2k$ on $SL_2(\mathbb{Z})$ and by S_{2k} the subspace of cusp forms. For a number field K we let

$$M_{2k}^0(K)=\left\{f\in M_{2k}\,|\,f(z)=\sum_{m\geqslant 0}a(m)\exp(2\pi imz)\text{ with }a(m)\in K\text{ for all }m\right\}$$

and

$$M_{2k}^{\pm}(K)=\{f\in M_{2k}\,|\,r_n(f)\in K\text{ for all }n\in\mathbb{Z}\text{ with }0\leqslant n\leqslant 2k-2,\,(-1)^n=\pm 1\}.$$

Here by definition

$$r_n(f)=\frac{n!}{(2\pi)^{n+1}}L(f,n+1)\quad(0\leqslant n\leqslant 2k-2),$$

where $L(f,s)$ denotes the L-function defined by meromorphic continuation of the Dirichlet series $\sum_{m\geqslant 1}a(m)m^{-s}$ ($\mathrm{Re}(s)>2k$; $a(m)=m$th Fourier coefficient of f). We put

$$S_{2k}^{0}=M_{2k}^{0}\cap S_{2k}$$

and

$$S_{2k}^{\pm}=M_{2k}^{\pm}\cap S_{2k}.$$

Note that if f is a cusp form, then

$$r_n(f)=\int_0^\infty f(it)t^n\,\mathrm{d}t,$$

so $r_n(f)$ is a usual period integral of f. On the other hand, for the Eisenstein series

$$G_{2k}(z)=-\frac{B_{2k}}{2k}+\sum_{m\geqslant 1}\sigma_{2k-1}(m)\exp(2\pi imz)$$

$$\left(B_{2k}=2k\text{-th Bernoulli number, }\sigma_{2k-1}(m)=\sum_{d\mid m}d^{2k-1}\right),$$

we have

$$r_n(G_{2k})=0\quad(0<n<2k-2,\ n\text{ even}),$$

$$r_0(G_{2k})=\frac{(-1)^{k-1}(2k-2)!}{2^{2k}}\cdot\frac{\zeta(2k-1)}{\pi^{2k-1}},$$

$$r_{2k-2}(G_{2k})=(-1)^k r_0(G_{2k}),$$

$$r_n(G_{2k})=(-1)^{(n+1)/2}\cdot\frac{1}{2}\cdot\frac{B_{n+1}}{n+1}\cdot\frac{B_{2k-1-n}}{2k-1-n}\quad(0<n<2k-2,\ n\text{ odd})$$

([3], §4·1). These formulas follow easily from the identity $L(G_{2k},s)=\zeta(s)\zeta(s-2k+1)$, the functional equation of $\zeta(s)$ and the fact that $\zeta(1-n)=-B_n/n$ ($n\geqslant 2$ even). Notice that the periods $r_n(f)$ are normalized in a natural way to make the spaces $S_{2k}^{+}(K)$ and $S_{2k}^{-}(K)$ dual to each other w.r.t. the usual Petersson scalar product ([3], §1).

The K-vector spaces $M_{2k}^{0}(K)$ and $M_{2k}^{\pm}(K)$ define K-rational structures on M_{2k}, i.e.

$$M_{2k}^{0}(K)\otimes_K\mathbb{C}\cong M_{2k}$$

and

$$M_{2k}^{\pm}(K)\otimes_K\mathbb{C}\cong M_{2k}.$$

Moreover,

$$M_{2k}^{0}(K)=KG_{2k}\oplus S_{2k}^{0}(K),$$

$$M_{2k}^{-}(K)=KG_{2k}\oplus S_{2k}^{-}(K)$$

and

$$M_{2k}^{+}(K)=Kc^{-1}G_{2k}\oplus S_{2k}^{+}(K)$$

where

$$\zeta(2k-1)$$

Conjecture. Let K be a number field. Then $M_{2k}^{+}(K) \cap M_{2k}^{0}(K) = \{0\}$ and $S_{2k}^{-}(K) \cap S_{2k}^{0}(K) = \{0\}$.

This conjecture seems plausible both from an empirical point of view–to the author's knowledge nobody so far has found any natural example of a non-zero function in the intersection of the above spaces – and also from a theoretical point of view, since e.g. the periods of a normalized primitive Hecke eigenform of weight 2 and level N are known to be either zero or transcendental ([1], §3).

Let $f \in S_{2k}$ be a normalized Hecke eigenform. Then its Fourier coefficients are algebraic numbers. Also by a result of Manin ([3]) there exist non-zero real numbers $\omega_{\pm}(f)$ such that the ratios $r_n(f)/\omega_{\pm}(f)$ $(0 \leqslant n \leqslant 2k-2, (-1)^n = \pm 1)$ are algebraic and in fact contained in the number field generated over $\mathbb{Q}$ by the Fourier coefficients of f. Hence

COROLLARY 1 TO THE CONJECTURE

Let $f \in S_{2k}$ be a normalized Hecke eigenform. Then the periods $r_0(f), r_1(f), \ldots, r_{2k-2}(f)$ are transcendental.

On the other hand, if we apply the conjecture to the Eisenstein series G_{2k} we obtain.

COROLLARY 2 TO THE CONJECTURE

Let $\zeta(s)$ be the Riemann zeta function. Then the numbers $\zeta(2k-1)/\pi^{2k-1}$ $(k = 2, 4, 6, \ldots)$ are transcendental.

References

[1] Bertrand D, Varietes abeliennes et formes lineaires d' integrales elliptiques, in: *Sém. de Théorie des Nombres*, Paris 1979–80 (ed. M.-J. Bertin), pp. 15–27, *Progress in Maths*. **12** Birkhäuser, Boston-Basel-Stuttgart 1981

[2] Beukers F, Irrationality proofs using modular forms, *Soc. Math. de France*, Astérisque 147–148 (1987), pp. 271–283

[3] Kohnen W and Zagier D, Modular forms with rational periods, in: *Modular Forms* (ed. R A Rankin), pp. 197–249, (Chichester: Ellis Horwood) 1984.

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On Shintani correspondence

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Abstract. For an even natural number m we will construct a Shintani lifting from $S_{2k}(m, \psi^2)$ to $S_{k+1/2}(\Gamma_0(2m), \psi_0)$, ψ a Dirichlet character modulo $2m$, $\psi_0 = \left(\frac{\psi(-1)}{\cdot}\right)\psi$, which is adjoint to the Shimura lifting. When m is squarefree we will combine this result with the multiplicity 1 theorem proved in [3] to give a formula for the product $c(m_1)\overline{c(m_2)}$ (m_2 squarefree) of two Fourier coefficients of a new form in $S^{\text{new}}_{k+1/2}(\Gamma_0(2m))$ in terms of certain cycle integrals of the corresponding new form in $S^{\text{new}}_{2k}(m)$. If $m_1 = m_2$ we will get Waldspurger's result expressing the square $c(m_1)^2$ in terms of an L-series of f at the centre of the critical strip. Similar product formulas for the Fourier coefficients of a Hecke-Pizer eigenform in $S^{\text{old}}_{k+1/2}(\Gamma_0(2m))$ and $S^{+,\text{old}}_{k+1/2}(\Gamma_0(2m))$ and of a Hecke eigenform in $S^{+,\text{old}}_{k+1/2}(\Gamma_0(4q))$ ($q \equiv 3(4)$ is a prime) are also given.

Keywords. Shimura lifting; Shintani lifting; kernel function; Waldspurger result.

1. Introduction

Shintani [9] constructed a lifting map from $S_{2k}(m, \psi^2)$, the space of cusp forms of weight $2k$ on $\Gamma_0(m)$ and character ψ^2 to $S_{k+1/2}(\Gamma_0(4m), \psi')$, the space of cusp forms of weight $k + \frac{1}{2}$ on $\Gamma_0(4m)$ and character ψ', where $k, m \in \mathbb{N}$, ψ a Dirichlet character modulo m, $\psi' = \left(\frac{m}{\cdot}\right)\left(\frac{-1}{\cdot}\right)^k$. When m is odd, Kohnen ([1, 2]) constructed a similar lifting from $S_{2k}(m)$ to his $+$space $S^{+}_{k+1/2}(\Gamma_0(4m))$, which is adjoint to the Shimura-Kohnen lifting. He combined this lifting with the "multiplicity 1" theorem proved in [1] to give a formula for the product $c(m_1)\,\overline{c(m_2)}$ of two arbitrary Fourier coefficients of a Hecke eigenform g in $S^{+,\text{new}}_{k+1/2}(\Gamma_0(4m))$, m squarefree, in terms of certain cycle integrals of the corresponding form f in $S^{\text{new}}_{2k}(m)$. In particular, if $m_1 = m_2$, one gets back Waldspurger's results expressing the square $c(m_1)^2$ in terms of an L-series of f at the centre of the critical strip.

In this paper, we prove that if m is even one can get a lifting map from $S_{2k}(m, \psi^2)$ to $S_{k+1/2}(\Gamma_0(2m), \psi_0)$, where ψ is a Dirichlet character modulo $2m$, $\psi_0 = \left(\frac{\psi(-1)}{\cdot}\right)\cdot\psi$, with some restriction on the even part of the conductor of ψ, which is adjoint to the Shimura lifting. As done by Kohnen, when m is even squarefree, we combine the above result with the "multiplicity 1" theorem proved in [3] to give a formula for the product $c(m_1)\,\overline{c(m_2)}$ (m_2 squarefree) of two Fourier coefficients of a newform g in $S^{\text{new}}_{k+1/2}(\Gamma_0(2m))$ in terms of certain cycle integrals of the corresponding newform f in

$S_{2k}^{\mathrm{new}}(m)$. In particular, when $m_1 = m_2$, one gets back Waldspurger's results expressi the square $c(m_1)^2$ in terms of an L-series of f at the centre of the critical strip.

Let m be odd and squarefree. In [4], we introduced analogous Pizer operat $C(p^2)$ in $S_{k+1/2}(\Gamma_0(4m))$ (resp. $S_{k+1/2}^{+}(\Gamma_0(4m))$) for primes $p|2m$ (resp. $p|m$). We redef $C(4)$ on $S_{k+1/2}(\Gamma_0(4m))$ by

$$C(4) = U(4) + W(4)U(4)W(4) + 2^{k-1}W(4)$$

(The Atkin-Lehner involution $W(4)$ in $S_{k+1/2}(\Gamma_0(4m))$ and the operators $U(l)$, $l|N$ defined as in [3]). We call a form in $S_{k+1/2}(\Gamma_0(4m))$ (resp. $S_{k+1/2}^{+}(\Gamma_0(4m))$), a Hecke-Pi eigenform, if it is an eigenform of all Hecke operators $T(p^2), p\nmid 2m$ (resp. $p\nmid m$) Pizer operators $C(p^2)$, $p|2m$ (resp. $p|m$). Then using the above Shintani lifting and results of [2, 4, 5], we also get similar formulas for the product of Fourier coefficie of an oldform in $S_{k+1/2}(\Gamma_0(4m))$ (resp. $S_{k+1/2}^{+}(\Gamma_0(4m))$, which is a Hecke-Pizer eigenfo and of an oldform in $S_{k+1/2}^{+}(\Gamma_0(4q))$ ($q \equiv 3(4)$ is a prime), which is a Hecke eigenfo

2. Notation

Let $k, M \in \mathbb{N}$, M be odd. Put $N = 2^{\lambda}M$, $\lambda \in \mathbb{N}$. If $z \in \mathbb{C}^*$ and $x \in \mathbb{C}$ we put $z^x = e^{x}$ where $\log z = \log|z| + i \arg z$ and the argument is determined by $-\pi < \arg z \leqslant \pi$. integers x, y with $y \neq 0$, we write $e_y(x)$ instead of $e^{2\pi i x/y}$.

We use the extended definition of the symbol $\left(\frac{c}{d}\right)$ as in [2]. For $m \in \mathbb{N}$,

$\Gamma_0(m) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| m|\gamma \right\}$ and write $\Gamma(1)$ for $\mathrm{SL}_2(\mathbb{Z})$.

Let χ_1 be a primitive Dirichlet character modulo M_1, where $M_1|M$ and χ_2 b primitive Dirichlet character modulo $2^{a'}$ where

$$a' = \begin{cases} \lambda \text{ or } \lambda+1 & \lambda > 2;\ t \text{ odd} \\ \lambda+1 & \lambda \geqslant 3;\ t \text{ even} \\ 0 \text{ or } 2 & \lambda = 2;\ t \text{ even} \\ 2 \text{ or } 3 & \lambda = 2;\ t \equiv 1(4) \\ 0 \text{ or } 3 & \lambda = 2;\ t \equiv 3(4) \\ 0 & \text{otherwise} \end{cases}$$

and t is a squarefree integer such that $\varepsilon(-1)^k t > 0$, $(t, M_1) = 1$ with $\varepsilon = \chi(-1)$, $\chi = \chi_1$

Put $\chi_0 = \left(\frac{4\varepsilon}{\cdot}\right)\chi$.

Let $S_{k+1/2}^{+}(\Gamma_0(4M), \chi_0)$ denote the Kohnen+space of $S_{k+1/2}(\Gamma_0(4M), \chi_0)$, consis of forms whose nth Fourier coefficient vanishes whenever $\varepsilon(-1)^k n \equiv 2, 3(4)$ ([1]).

$$D = \begin{cases} t & t \equiv 1(4) \\ 4t & t \equiv 2, 3(4) \end{cases}; \quad D_0 = \begin{cases} t & \lambda \geqslant 2 \\ D & \lambda = 1 \end{cases};$$

$$l = \begin{cases} 1 & \lambda \geqslant 2 \\ 4 & \lambda = 1 \end{cases}; \quad M_2 = \begin{cases} 2M/M_1 & \lambda = 1;\ t \equiv 1(4) \\ M/M_1 & \text{otherwise} \end{cases}$$

Note that D is a fundamental discriminant with $\varepsilon(-1)^k D>0$. Let $\Delta=2^{b'}M_1^2\,|t|$, where

$$b'=\begin{cases} 2\lambda+2 & \lambda\geqslant 2;\ t \text{ odd, } \chi_2 \text{ primitive mod } 2^{\lambda+1} \\ & \text{or } \lambda=1;\ t=2,3(4) \\ 2\lambda & \text{otherwise} \end{cases}$$

For $m\in\mathbb{N}$, let $Q_{N,\Delta,m}$ be the set of all integral binary quadratic forms $[a,b,c]$ with discriminant $b^2-4ac=\Delta m$ and $a\equiv 0(2^{c'}MM_1)$, where

$$c'=\begin{cases} 2\lambda & \lambda\geqslant 2;\ t \text{ odd, } \chi_2 \text{ primitive mod } 2^{\lambda+1} \\ & \text{or } \lambda=1;\ t\equiv 2,3,(4) \\ 2\lambda-1 & \text{otherwise} \end{cases}$$

If f and g are cusp forms of half-integral weight on some subgroup Γ of finite index in $\Gamma(1)$, we denote by

$$\langle f,g\rangle=\frac{1}{[\Gamma(1):\Gamma]}\int_{\Gamma\backslash\mathscr{H}} f(z)\overline{g(z)}y^{k-3/2}\,\mathrm{d}x\,\mathrm{d}y$$

their Petersson product, where $z=x+iy$ belongs to the upper half-plane $\mathscr{H}$.

3. Statement of results

Let $\Gamma(1)$ act on integral binary quadratic forms $[a,b,c]\cdot(x,y)=ax^2+bxy+cy^2$ by

$$[a,b,c]\cdot\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}(x,y)=[a,b,c](\alpha x+\beta y,\gamma x+\delta y) \tag{2}$$

For a form $Q=[a,b,c]$ whose discriminant $|Q|$ is divisible by D such that $|Q|/D$ is a square modulo 4, we put

$$\omega_D(Q)=\begin{cases}\left(\dfrac{D}{r}\right) & \text{if } (a,b,c,D)=1 \text{ where } Q \text{ represents } r\\ 0 & \text{otherwise}\end{cases}$$

Note that $\omega_D(Q)$ is well-defined and depends only on $\Gamma(1)$ equivalence class of Q. For $m\in\mathbb{N}$, $k>2$, $z\in\mathscr{H}$ define

$$F_{k,N,\chi}(z;\Delta m)=\sum_{\substack{Q\in Q_{N,\Delta,m}\\ Q=[a,b,c]}}\chi(c)\,\omega_D(Q)Q(z,1)^{-k}. \tag{3}$$

This series converges absolutely and uniformly on compact sets and $F_{k,N,\chi}(z;\Delta m)\in S_{2k}(N,\bar{\chi}^2)$. For $k=1$, the series no longer converges absolutely but we define

$$F_{1,N,\chi}(z;\Delta m)=\lim_{s\to 0}\sum_{Q\in Q_{N,\Delta,m}}\chi(c)\frac{\omega_D(Q)}{Q(z,1)}\frac{\operatorname{Im}(z)^s}{|Q(z,1)|^s}\ (\operatorname{Re}s>0); \tag{4}$$

this belongs to $S_2(N,\bar{\chi}^2)$ when N is cubefree (cf. [2]).

Remark 1. In view of the above fact, we assume that N is cubefree when $k=1$.

For $k \geqslant 1$; z, $\tau \in \mathscr{H}$, put

$$\Omega_{k,N,\chi}(z,\tau;D) = i_N c_{k,D_0}^{-1}(2^{d'}M_1)^{k-\frac{1}{2}} \sum_{m\geqslant 1} (lm)^{k-\frac{1}{2}} \times \left(\sum_{r|M_2} \mu(r)\bar{\chi}(r)\left(\frac{D}{r}\right) r^{k-1} F_{k,N/r,\chi}(rz;\Delta m)\right) e^{2\pi i m\tau} \tag{5}$$

where

$$i_N = [\Gamma(1):\Gamma_0(N)];\, d' = \begin{cases} \lambda+1 & \lambda \geqslant 2;\, t \text{ odd}, \chi_2 \text{ primitive mod } 2^{\lambda+1} \\ 0 & \lambda = 1 \\ \lambda & \text{otherwise} \end{cases}$$

and

$$c_{k,D_0} = (-1)^{[k/2]}|D_0|^{-k+1/2}\pi \binom{2k-2}{k-1} 2^{-3k+\beta}$$

with

$$\beta = \begin{cases} 3 & \lambda \geqslant 2 \\ 4 & \lambda = 1;\, t \equiv 1(4) \\ 5 & \lambda = 1;\, t \equiv 2,3(4) \end{cases}$$

For $n \in \mathbb{N}$, the n-th Poincaré series in $S_{k+1/2}(\Gamma_0(2^{e'}N),\chi_0)$ is given as follows. For $k \geqslant 2$

$$P_{k,2^{e'}N,n,\chi_0}(\tau) = \tfrac{1}{2} \sum_{A \in \Gamma_\infty \backslash \Gamma_0(2^{e'}N)} \bar{\chi}_0(d) j(A,\tau)^{-2k-1} e(nA\tau)$$

and for $k=1$,

$$P_{1,2^{e'}N,n,\chi_0}(\tau) = \lim_{s\to 0} \tfrac{1}{2} \sum_{A \in \Gamma_\infty \backslash \Gamma_0(2^{e'}N)} \bar{\chi}_0(d) j(A,\tau)^{-3} |j(A,\tau)|^{-4s} e(nA\tau),$$

where

$$e' = \begin{cases} 1 & \lambda \geqslant 2 \\ 2 & \lambda = 1;\, t \equiv 1(4) \\ 3 & \lambda = 1;\, t \equiv 2,3(4) \end{cases}$$

and

$$j(A,\tau) = \left(\frac{c}{d}\right)\left(\frac{-4}{d}\right)^{-\frac{1}{2}} (c\tau+d)^{1/2} \quad \text{for } A = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0(2^{e'}N)$$

Then, we have

$$\langle f, P_{k,2^{e'}N,n,\chi_0} \rangle = i_{2^{e'}N}^{-1} \frac{\Gamma(k-\frac{1}{2})}{(4\pi n)^{k-\frac{1}{2}}} a_f(n) \tag{6}$$

for all $f \in S_{k+1/2}(\Gamma_0(2^{e'}N),\chi_0)$, where $a_f(n)$ denotes the n-th Fourier coefficient of f.

Theorem 1. *The function $\Omega_{k,N,\chi}(z,\tau;D)$ defined by* (5) *has the Fourier expansion*

$$\Omega_{k,N,\chi}(z,\tau;D) = \frac{i_N c_{k,D_0}^{-1}(-1)^{[k/2]} 2(2\pi)^k}{(k-1)!} R_{\chi,t} \times \sum_{n\geqslant 1} n^{k-1} \left(\sum_{\substack{d|n \\ (d,M_2)=1}} \bar{\chi}(d)\left(\frac{D}{d}\right)(n/d)^k \right) \times P_{k,2^{e'}N,(n^2|D_0|/d^2),\chi_0} | U(l)) e^{2\pi i n z} \tag{7}$$

with respect to z, where $R_{\chi,t}$ *is the Gauss sum given by*

$$R_{\chi,t}=(2^{d'}M_1|D_0|)^{-\frac{1}{2}}\left(\frac{4\varepsilon t}{-1}\right)^{-\frac{1}{2}}\sum_{r(2^{d'}M_1|D_0|)}\chi(r)\left(\frac{D}{r}\right)e_{2^{d'}M_1|D_0|}(r)$$

In particular, for fixed $z\in\mathscr{H}$, *it belongs to* $S_{k+1/2}(\Gamma_0(2N),\chi_0)$ *with respect to* τ.

For $f\in S_{k+1/2}(\Gamma_0(2N),\chi_0)$, the t-th Shimura lifting is defined by

$$f\Big|\mathscr{S}_{t,k,N,\chi}=\sum_{n\geqslant 1}\Bigg(\sum_{\substack{d|n\\(d,M_2)=1}}\chi(d)\left(\frac{4t}{d}\right)d^{k-1}a_f(n^2|t|/d^2)\Bigg)e^{2\pi inz} \tag{8}$$

Put

$$\mathscr{S}_{t,\lambda}=\begin{cases}\mathscr{S}_{t,k,N,\chi} & \lambda\geqslant 2\\ W(4)U(4)W(4)\mathscr{S}_{t,k,N,\chi} & \lambda=1;\ t\equiv 1(4)\\ (W(4)U(4))^2\mathscr{S}_{t,k,N,\chi} & \lambda=1;\ t\equiv 2,3(4)\end{cases} \tag{9}$$

For a form $F\in S_{2k}(N,\chi^2)$, set

$$r_{k,N,\chi}(F;\Delta m)=\sum_{\substack{Q\bmod\Gamma_0(N)\\|Q|=\Delta m\\a\equiv 0(2^{c'}MM_1)}}\chi(c)\omega_D(Q)\int_{C_Q}F(z)d_{Q,k}z \tag{10}$$

where C_Q is the image in $\Gamma_0(N)\backslash\mathscr{H}$ of the semicircle $a|z|^2+b\operatorname{Re}z+c=0$ oriented from $((-b-\sqrt{\Delta m})/2a)$ to $((-b+\sqrt{\Delta m})/2a)$, if $a\neq 0$ or of the vertical line $b\operatorname{Re}z+c=0$ oriented from $-c/b$ to $i\infty$ if $b>0$ and from $i\infty$ to $-c/b$ if $b<0$, if $a=0$, and

$$d_{Q,k}z=\begin{cases}(az^2+bz+c)^{k-1}\,dz & \chi^2=1\\ (az^2-bz+c)^{k-1}\,dz & \chi^2\neq 1\end{cases}$$

Now, for $F\in S_{2k}(N,\chi^2)$, define

$$F\Big|\mathscr{S}^*_{t,\lambda}=(-1)^{[k/2]}2^{k-\beta+2}(2^{d'}M_1)^{-k+1/2}\bar{R}_{\chi,t}$$
$$\times\sum_{m\geqslant 1}\Bigg(\sum_{r|M_2}\mu(r)\bar{\chi}(r)\left(\frac{D}{r}\right)r^{-k}r_{k,Nr,\chi}(F;\Delta mr^2)\Bigg)e^{2\pi im\tau} \tag{11}$$

Theorem 2. *Let* $f\in S_{k+1/2}(\Gamma_0(2N),\chi_0)$ *and* $F\in S_{2k}(N,\chi^2)$. *Then*

$$R_{\chi,t}\langle f,\Omega_{k,N,\chi}(-\bar{z},\tau;D)\rangle=f|\mathscr{S}_{t,\lambda} \tag{12}$$

$$\bar{R}_{\chi,t}\langle F,\overline{\Omega_{k,N,\chi}(-\bar{z},\tau;D)}\rangle=F|\mathscr{S}^*_{t,\lambda} \tag{13}$$

Theorem 3. *Define*

$$\mathscr{S}^*_{t,k,N,\chi}=\begin{cases}\mathscr{S}^*_{t,\lambda} & \lambda\geqslant 2\\ \mathscr{S}^*_{t,\lambda}(2^{-2k+1}W(4)U(4)W(4)-2^{-k}W(4)) & \lambda=1;\ t\equiv 1(4)\\ \mathscr{S}^*_{t,\lambda}(3\cdot 2^{-2k}-2^{-3k+1}W(4)U(4)) & \lambda=1;\ t\equiv 2,3(4)\end{cases} \tag{14}$$

Then

$$\mathscr{S}^{*}_{t,k,N,\chi}\colon S_{2k}(N,\chi^2)\to S_{k+1/2}(\Gamma_0(2N),\chi_0)$$

is adjoint to the t-th Shimura lifting $\mathscr{S}_{t,k,N,\chi}$ *with respect to the Petersson scalar product.*

In the remaining part of this section we assume that M is square-free and $\lambda = M_1 = 1$.

Theorem 4. *Let* $m,n\in\mathbb{N}$, *n be squarefree. Let* $f\in S^{\text{new}}_{k+1/2}(\Gamma_0(4M))$ *be a Hecke eigenform and let* $F\in S^{\text{new}}_{2k}(2M)$ *be the corresponding Hecke eigenform under Shimura map with* $a_F(1)=1$. *Suppose that* $F|W_2 = \left(\frac{(-1)^k n}{2}\right)F$, *if* $(-1)^k n\equiv 1(4)$, *where* W_p, $p|2M$ *is the Atkin-Lehner involution in* $S_{2k}(2M)$. *Then, we have*

$$\frac{a_f(m)\overline{a_f(n)}}{\langle f,f\rangle}=\begin{cases}\dfrac{(-1)^{[k/2]+1}}{\langle F,F\rangle}2^{-1}\left(\dfrac{(-1)^k n}{2}\right)r_{k,N}(F;4mn) & (-1)^k n\equiv 1(4)\\[2ex] \dfrac{(-1)^{[k/2]}}{\langle F,F\rangle}2^{-k-1}r_{k,N}(F;16mn) & (-1)^k n\equiv 2,3(4)\end{cases}\tag{15}$$

Remark 2. Let $f\in S^{\text{old}}_{k+1/2}(\Gamma_0(4M))$ and $F\in S^{\text{old}}_{2k}(2M)$ be Hecke-Pizer eigenforms under Shimura correspondence with $a_F(1)=1$. Suppose that $F|W_2=\omega_2 F$, $\omega_2\in\{\pm 1\}$. Then, for $n\in\mathbb{N}$ with $(-1)^k n\equiv 2,3(4)$, n squarefree we have

$$\frac{a_f(m)\overline{a_f(n)}}{\langle f,f\rangle}=3\frac{(-1)^{[k/2]}}{\langle F,F\rangle}2^{-k-3}\sum_{r|M}\mu(r)\left(\frac{(-1)^k 4n}{r}\right)r^{-k}\left\{r_{k,Nr}(F;16mnr^2)-\frac{2^{-k+1}}{3}\omega_2 r_{k,Nr}(F|U(2);16mnr^2)\right\}\tag{16}$$

Theorem 5. *Let* $f\in S^{+,\text{old}}_{k+1/2}(\Gamma_0(4m))$ *be a Hecke-Pizer eigenform and* $F\in S^{\text{old}}_{2k}(M)$ *be the corresponding normalized form via Shimura-Kohnen map. Let* $m,n\in\mathbb{N}$ *with* $(-1)^k m\equiv$ 0, 1(4) *and* $(-1)^k n$, *a fundamental discriminant. Then*

$$\frac{a_f(m)\overline{a_f(n)}}{\langle f,f\rangle}=\frac{(-1)^{[k/2]}2^k}{\langle F,F\rangle}\sum_{r|M}\mu(r)\left(\frac{(-1)^k n}{r}\right)r^{-k}r_{k,Mr}(F;mnr^2)\tag{17}$$

Theorem 6. *Let* $q\equiv 3(4)$ *be a prime. Let* $f\in S^{+,\text{old}}_{k+1/2}(\Gamma_0(4q))$ *be a Hecke eigenform and* $F\in S^{\text{old}}_{2k}(q)$ *be the corresponding normalized form under Shimura-Kohnen lifting. Then for* $m,n\in\mathbb{N}$ *with* $(-1)^k m\equiv 0,1(4)$ *and* $(-1)^k n$, *a fundamental discriminant,*

$$\frac{a_f(m)\overline{a_f(n)}}{\langle f,f\rangle}=\frac{(-1)^{[k/2]}2^k}{\langle F,F\rangle}\sum_{r|q}\mu(r)\left(\frac{(-1)^k n}{r}\right)r^{-k}r_{k,qr}(F;mnr^2)\tag{18}$$

Remark 3. In Theorems 5 and 6, the cycle integral $r_{k,.}(F;.)$ is the same as defined in ([2], eq 7).

For a fundamental discriminat D with $(D,N)=1$ we denote by

$$L(F,D,s)=\sum_{n\geqslant 1}\left(\frac{D}{n}\right)a_F(n)n^{-s}\quad(\operatorname{Re}(s)\gg 0)$$

the L-series of $F \in S_{2k}(N)$ twisted by the quadratic character $\left(\frac{D}{\cdot}\right)$.

Using Theorem 4 one can deduce the following corollaries in a similar way as in [2].

COROLLARY 1.

Let $n \in \mathbb{N}$ be squarefree and $D = (-1)^k n \equiv 1(4)$ such that $(D, M) = 1$. The functions f and F be as in Theorem 4. Suppose that for all primes $p \mid N$ we have $F \mid W_p = \left(\frac{D}{p}\right) F$. Then

$$\frac{|a_f(n)|^2}{\langle f, f \rangle} = 2^{\nu(M)-1} \frac{(k-1)!}{\pi^k} n^{k-\frac{1}{2}} \frac{L(F, D, k)}{\langle F, F \rangle} \tag{19}$$

where $\nu(M)$ denotes the number of distinct prime divisors of M.

COROLLARY 2.

Let $D \equiv D' \equiv 1(4)$ be two fundamental discriminants which are relatively prime to M. Then

$$(DD')^{k-\frac{1}{2}} L(F, D, k) L(F, D', k) = \frac{\pi^{2k}}{(k-1)!^2} 2^{-2\nu(M)} |r_{k,N}(F; 4DD')|^2 \tag{20}$$

4. Proofs of theorems

Since the method of proofs of Theorems is similar as in [2], we only sketch the proofs.

4.1 *Proof of theorem* 1

For the proof of Theorem 1 we need the following propositions which can be proved as done in [2].

PROPOSITION 1.

The function $F_{k,N,\chi}(z; \Delta m)$ has the expansion

$$F_{k,N,\chi}(z; \Delta m) = \sum_{n \geqslant 1} c_{k,N,\chi}(n; \Delta m) e^{2\pi i n z} \tag{21}$$

where

$$\begin{aligned} c_{k,N,\chi}(n; \Delta m) = {} & \frac{2(-2\pi)^k}{(k-1)!} (2^{d'} M_1)^{-k+1/2} (n^2/lm|D_0|)^{(k-1)/2} \\ & \times \Bigg\{ (-1)^{[(k+1)/2]} R_{\gamma,t} \bar{\chi}(n/(lm/D_0)^{1/2}) \left(\frac{D}{n/(lm/|D_0|)^{1/2}} \right) \delta \\ & \times \left(\frac{n}{(lm/|D_0|)^{1/2}} \right) |D_0|^{-\frac{1}{2}} + \pi \sqrt{2} (n^2/ln|D_0|)^{1/4} \\ & \times \sum_{\substack{a \geqslant 1 \\ 2^{c'} M M_1 | a}} a^{-\frac{1}{2}} S_{a,\chi}(\Delta m, n) J_{k-\frac{1}{2}} \left(\frac{\pi n \sqrt{\Delta m}}{a} \right) \Bigg\}; \end{aligned}$$

here

$$\delta(x) = \begin{cases} 1 & x \in \mathbb{Z} \\ 0 & x \notin \mathbb{Z} \end{cases}$$

$$S_{a,\chi}(\Delta m, n) = \sum_{\substack{b(2a) \\ b^2 \equiv \Delta m(4a)}} \chi\left(\frac{b^2 - \Delta m}{4a}\right) \omega_D\left(a, b, \frac{b^2 - \Delta m}{4a}\right) e_{2a}(nb) \tag{22}$$

is a finite exponential sum and $J_{k-\frac{1}{2}}(.)$ *is the Bessel function of order* $k - \frac{1}{2}$.

PROPOSITION 2.

$$P_{k,2^{e'}N,n,\chi_0}(\tau) = \sum_{m \geqslant 1} g_{k,2^{e'}N,n,\chi_0}(m) e^{2\pi i m \tau} \tag{23}$$

where

$$g_{k,2^{e'}N,n,\chi_0}(m) = \delta_{n,m} + \pi\sqrt{2}(-1)^{[(k+1)/2]}(1 - (-1)^k i)(m/n)^{(k/2)-(1/4)}$$
$$\times \sum_{c \geqslant 1} H_{2^{e'}Nc,\chi}(m,n) J_{k-\frac{1}{2}}\left(\frac{\pi\sqrt{mn}}{2^{e'-2}Nc}\right);$$

here $\delta_{n,m}$ *is the Kronecker delta and*

$$H_{2^{e'}Nc,\chi}(m,n) = \frac{1}{2^{e'}Nc} \sum_{\delta(2^{e'}Nc)} \bar{\chi}_0(\delta)\left(\frac{2^{e'}Nc}{\delta}\right)\left(\frac{-4}{\delta}\right)^{k+1/2} e_{2^{e'}Nc}(m\delta + n\delta^{-1})$$
$$(\delta^{-1} \in \mathbb{Z};\ \delta\delta^{-1} \equiv 1(2^{e'}Nc)) \tag{24}$$

Note that when $\lambda = 1$, $U(4)$ maps the space $S_{k+1/2}(\Gamma_0(2^e N), \chi_0)$ to the space $S_{k+1/2}(\Gamma_0(2N), \chi_0)$. This shows that the function $\Omega_{k,N,\chi}(z, .; D)$ belongs to $S_{k+1/2}(\Gamma_0(2N), \chi_0)$. We will now prove (7). Using (21) into the left-hand side of (7) and (23) into the right-hand side of (7), we see that for all $m, n \geqslant 1$, we have to prove that

$$(-1)^{[k/2]} R_{\chi,t}(\mathrm{lm}/|D_0|)^{k/2} \bar{\chi}(n/(\mathrm{lm}/|D_0|)^{1/2}) \left(\frac{D}{n/(\mathrm{lm}/|D_0|)^{1/2}}\right) \sum_{r|M_2} \mu(r)$$
$$\times\ \delta(n/(r(\mathrm{lm}/|D_0|)^{1/2})) + \pi\sqrt{2}(-1)^k (\mathrm{lm}/|D_0|)^{(k/2)-(1/4)} n^{1/2} \sum_{a \geqslant 1} (2^{c'}MM_1 a)^{-1/2}$$
$$\times \sum_{r|M_2} \mu(r)\bar{\chi}(r)\left(\frac{D}{r}\right) S_{2^{c'}MM_1 a/r,\chi}\left(\Delta m, \frac{n}{r}\right) J_{k-1/2}\left(\frac{\pi n\sqrt{\Delta m}}{2^{c'}MM_1 a}\right)$$
$$= (-1)^{[k/2]} R_{\chi,t} \sum_{\substack{d|n \\ (d,M_2)=1}} \bar{\chi}(d)\left(\frac{D}{d}\right)(n/d)^k\, \delta_{\mathrm{lm},n^2(|D_0|/d^2)} + \pi\sqrt{2}(-1)^k$$
$$\times (1 - (-1)^k i) R_{\chi,t}(\mathrm{lm}/|D_0|)^{(k/2)-(1/4)} n^{1/2}$$
$$\times \sum_{\substack{d|n \\ (d,M_2)=1}} \bar{\chi}(d)\left(\frac{D}{d}\right) d^{-1/2} \sum_{c \geqslant 1} H_{2^{e'}Nc,\chi}(\mathrm{lm}, n^2|D_0|/d^2)$$
$$\times J_{k-1/2}\left(\frac{\pi n\sqrt{\mathrm{lm}|D_0|}}{2^{e'-2}Ncd}\right). \tag{25}$$

One can easily check the equality of the first terms of (25). Putting $cd = a$ on the righ

hand side of (25), we see that for the proof of (7), it is sufficient to prove the following

PROPOSITION 3.

For all $a, m, n \in \mathbb{N}$ with $M_1 | a$,

$$S_{2^{c'}M_1 a,\chi}(\Delta m, n) = (2^{c'}M_1)^{1/2} R_{\chi,t}(1-(-1)^k i) \sum_{d|(a,n)} \bar{\chi}(d)\left(\frac{D}{d}\right)(a/d)^{1/2} \times H_{2^{c'+\lambda}a/d,\chi}\left(lm, \frac{n^2|D_0|}{d^2}\right) \tag{26}$$

Proof. As functions of n, both sides are periodic with period $2^{r'}a$, where

$$r' = \begin{cases} 2 & \lambda = 1 \\ \lambda & \lambda \geqslant 2 \end{cases}$$

So it will be sufficient to show that for every $h \in \mathbb{Z}$,

$$\begin{aligned} &\frac{1}{2^{r'}a} \sum_{n(2^{r'}a)} e_{2^{r'}a}(-hn) S_{2^{c'}M_1 a,\chi}(\Delta m, n) \\ &= \frac{(2^{c'}M_1)^{1/2}}{2^{r'}a} R_{\chi,t}(1-(-1)^k i) \sum_{n(2^{r'}a)} e_{2^{r'}a}(-hn) \\ &\times \sum_{d|(a,n)} \bar{\chi}(d)\left(\frac{D}{d}\right)(a/d)^{1/2} H_{2^{c'+\lambda}a/d,\chi}\left(lm, \frac{n^2|D_0|}{d^2}\right) \end{aligned} \tag{27}$$

Using Proposition 6 of [2],

$$\text{LHS of (27)} = \begin{cases} \chi\left(\frac{\Delta_1}{2^{s'}a/M_1}\right) \prod_{p^v \| 2^{s'}a/M_1} \left(\frac{D/p^*}{p^v}\right)\left(\frac{P^*}{\Delta_1/p^v}\right) & \Delta_1 \equiv 0(2^{s'}a/M_1) \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

where

$$\Delta_1 = h_1^2 - lm|D_0|, \quad h_1 = \begin{cases} 2h & \lambda = 1;\ t \equiv 2, 3, (4) \\ h & \text{otherwise} \end{cases};$$

$$s' = \begin{cases} 0 & \lambda \geqslant 2;\ t \text{ odd, } \chi_2 \text{ primitive mod } 2^{\lambda+1} \\ 3 & \lambda = 1;\ t \equiv 1(4) \\ 4 & \lambda = 1;\ t \equiv 2, 3(4) \\ 1 & \text{otherwise} \end{cases}$$

and p^* is such that p^* and D/p^* are fundamental discriminants. Denote by $C_{D,m,\chi}(a, h)$, the right-hand side of (27). After making some suitable substitutions, we get

$$C_{D,m,\chi}(a, h) = \frac{\left(\frac{4\varepsilon}{-1}\right)^{1/2}(2^{c'}M_1)^{1/2}}{2^{r'}a} R_{\chi,t} \sum_{d|a} \bar{\chi}(d)\left(\frac{D}{d}\right)(a/d)^{1/2} \mathscr{F}_{2^{c'+\lambda}a/d,\chi}(Q),$$

where

$$\mathscr{F}_{2^{e'+\lambda}a/d,\chi}(Q) = \frac{1}{2^{e'+\lambda}a/d} \sum_{\substack{\delta | (2^{e'+\lambda}a/d) \\ n (2^{r'}a/d)}} \bar{\chi}(\delta) \left(\frac{2^{e'+\lambda}a/d}{\delta} \right)$$

$$\times \left(1 - \left(\frac{-4}{\delta} \right) i \right) e_{2^{e'+\lambda}a/d}(\delta Q(n))$$

with

$$Q(n) = D_0 n^2 - 2h_1 n + \varepsilon(-1)^k l m.$$

After similar computations as done in [2], we obtain

$$C_{D,m,\chi}(a,h) = \begin{cases} \chi\left(\dfrac{\Delta_1}{2^{s'}a/M_1} \right) \displaystyle\prod_{p^v \| 2^{s'}a/M_1} \left(\dfrac{D/p^*}{p^v} \right) \left(\dfrac{p^*}{\Delta_1/p^v} \right) & \Delta_1 \equiv 0(2^{s'}a/M_1) \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

This completes the proof of Theorem 1.

4.2 *Proof of theorem* 2

For $f \in S_{k+1/2}(\Gamma_0(2N), \chi_0)$, define the D-th Shimura-Kohnen lifting by

$$f | \mathscr{S}_{D,k,N,\chi} = \sum_{n \geq 1} \left(\sum_{\substack{d|n \\ (d, M/M_1)=1}} \chi(d) \left(\frac{D}{d} \right) d^{k-1} a_f(n^2 |D|/d^2) \right) e^{2\pi i n z} \tag{30}$$

Then it is easily seen that when $D \equiv 0(4)$,

$$\mathscr{S}_{D,k,N,\chi} = \mathscr{S}_{t,k,N,\chi} U(2) = U(4) \mathscr{S}_{t,k,N,\chi} \tag{31}$$

Note that when $\lambda = 1$ and $f \in S_{k+1/2}(\Gamma_0(2N), \chi_0)$

$$\langle f, P_{k,2^{e'}N,n,\chi_0} | U(4) \rangle = \langle f | W(4) U(4) W(4), P_{k,2^{e'}N,n,\chi_0} \rangle \tag{32}$$

Then (12) can be proved easily using (5), (6), (9), (30), (31) and (32).

It is easily seen that

$$\sum_{r|M_2} \mu(r) \bar{\chi}(r) \left(\frac{D}{r} \right) r^{k-1} F_{k,N/r,\chi}(rz; \Delta m)$$

$$= \sum_{r|M_2} \mu(r) \bar{\chi}(r) \left(\frac{D}{r} \right) r^{k-1} F_{k,Nr,\chi}(z; \Delta m r^2) \tag{33}$$

Note that when $\chi^2 = 1$,

$$\overline{\Omega_{k,N,\chi}(-\bar{z}, \tau; D)} = \Omega_{k,N,\chi}(z, -\bar{\tau}; D) \tag{34}$$

Then using (33) and (34) in (5), we see that (13) follows from the following proposition which can be proved similar to proposition 7 of [2].

PROPOSITION 4.

For $F \in S_{2k}(N, \chi^2)$,

$$\langle F, \overline{F_{k,N,\chi}(-\bar{z};\Delta m)} \rangle = i_N^{-1}\pi \binom{2k-2}{k-1} 2^{-2k+2}(\Delta m)^{-k+1/2} r_{k,N,\chi}(F;\Delta m) \quad (35)$$

where $r_{k,N,\chi}(F;\Delta m)$ *is defined by* (10).

4.3 *Proof of theorem* 3

When $\lambda \geqslant 2$, Theorem 3 is nothing but Theorem 2. So let us assume that $\lambda = 1$.

We know that the adjoint of $U(4)$ in $S_{k+1/2}(\Gamma_0(2N), \chi_0)$ is $W(4)U(4)W(4)$.

On $S_{k+1/2}(\Gamma_0(2N), \chi_0)$, one can prove that

$$(W(4)U(4))^2 - 2^{k-1}W(4)U(4) - 2^{2k-1} = 0.$$

From this it follows that

$$U(4)^{-1} = 2^{-2k}(2W(4)U(4)W(4) - 2^k W(4)) \quad (36)$$

and

$$(W(4)U(4))^{-2} = 2^{-2k}(3 - 2^{-k+1}W(4)U(4)). \quad (37)$$

Let $t \equiv 1(4)$. Then

$$\mathscr{S}_{t,\lambda} = W(4)U(4)W(4)\mathscr{S}_{t,k,N,\chi}$$

Taking adjoint,

$$\mathscr{S}^*_{t,\lambda} = \mathscr{S}^*_{t,k,N,\chi}U(4)$$

Therefore by (36),

$$\mathscr{S}^*_{t,k,N,\chi} = \mathscr{S}^*_{t,\lambda}(2^{-2k+1}W(4)U(4)W(4) - 2^{-k}W(4))$$

Let $t \equiv 2, 3(4)$. Then

$$\mathscr{S}_{t,\lambda} = (W(4)U(4))^2\mathscr{S}_{t,k,N,\chi}$$

Therefore,

$$\mathscr{S}^*_{t,\lambda} = \mathscr{S}^*_{t,k,N,\chi}(W(4)U(4))^2$$

Hence by (37),

$$\mathscr{S}^*_{t,k,N,\chi} = \mathscr{S}^*_{t,\lambda}\,(3 \cdot 2^{-2k} - 2^{-3k+1}W(4)U(4))$$

This completes the Proof of Theorem 3.

4.4 *Proof of theorem* 4 *and remark* 2

Note that by ([12], Corollary p. 543), Theorem 2 of [3] and hence the results of [3, 4]

also hold for $k=1$. Write

$$\mathscr{S}_{(-1)^k n,k,N} = \mathscr{S}_{(-1)^k n}$$

and

$$\mathscr{S}^*_{(-1)^k n,k,N} = \mathscr{S}^*_{(-1)^k n}$$

Because of the "multiplicity 1" theorem proved in [3, 4], we have

$$f \mid \mathscr{S}_{(-1)^k n} = a_f(n) F$$

and

$$F \mid \mathscr{S}^*_{(-1)^k n} = \alpha f, \quad \alpha \in \mathbb{C}.$$

Therefore we get

$$\begin{aligned}\alpha a_f(m) \langle f, f \rangle &= a_f(m) \langle F \mid \mathscr{S}^*_{(-1)^k n}, f \rangle \\ &= a_f(m) \langle F, f \mid \mathscr{S}_{(-1)^k n} \rangle \\ &= a_f(m) \overline{a_f(n)} \langle F, F \rangle. \end{aligned} \tag{38}$$

Since $U(4) = -2^{k-1} W(4)$ on $S^{\text{new}}_{k+1/2}(\Gamma_0(2N))$ and

$$W(4)\mathscr{S}_{t,k,N} = \mathscr{S}_{t,k,N} W_2, \quad \text{if } t \equiv 2, 3(4)$$

We have using (9) and (11),

$$\alpha a_f(m) = \begin{cases} (-1)^{[k/2]+1} 2^{-1} \left(\dfrac{(-1)^k n}{2}\right) r_{k,N}(F; 4mn) \\ \qquad f \in S^{\text{new}}_{k+1/2}(\Gamma_0(4M));\ (-1)^k n \equiv 1(4) \\ (-1)^{[k/2]} 2^{-k-1} r_{k,N}(F; 16mn) \\ \qquad f \in S^{\text{new}}_{k+1/2}(\Gamma_0(4M));\ (-1)^k n \equiv 2, 3(4) \\ (-1)^{[k/2]} 2^{-k-3} \sum\limits_{r \mid M} \mu(r) \left(\dfrac{(-1)^k 4n}{r}\right) r^{-k} \{3 r_{k,Nr}(F; 16mnr^2) \\ \quad - 2^{-k+1} \omega_2 r_{k,Nr}(F \mid U(2); 16mnr^2)\} \\ \qquad f \in S^{\text{old}}_{k+1/2}(\Gamma_0(4M));\ (-1)^k n \equiv 2, 3(4) \end{cases} \tag{39}$$

Here we have used the fact that

$$\langle F, F_{k,N/r}(rz; \Delta m) \rangle = 0 \text{ if } r > 1, \text{ when } F \in S^{\text{new}}_{2k}(2M).$$

Combining (38) and (39), we get the required results.

4.5 *Proof of theorems* 5 *and* 6

In [4] (resp. [5]), we have shown that $S^+_{k+1/2}(\Gamma_0(4M))$ (resp. $S^+_{k+1/2}(\Gamma_0(4q))$ can decomposed as a direct sum of one dimensional Hecke-Pizer eigenforms (resp. Hec eigenforms) and each such form corresponds to a unique form in $S_{2k}(M)$ (resp. $S_{2k}($ under Shimura-Kohnen map. Therefore, one can proceed as in [2] to obtain (and (18).

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References

[1] Kohnen W, Newforms of half-integral weight, *J. reine Angew Math.* **333** (1982) 32–72
[2] Kohnen W, Fourier coefficients of modular forms of half-integral weight, *Mathematische Annalen* **271** (1985) 237–268
[3] Manickam M, Ramakrishnan B, Vasudevan T C, On the theory of newforms of half-integral weight (1988) Preprint
[4] Manickam M, Ramakrishnan B, Vasudevan T C, Diagonalising modular forms of half-integral weight (1988) Preprint
[5] Manickam M, Ramakrishnan B, Vasudevan T C. On a remark of Atkin-Lehner, (1988) Preprint
[6] Niwa S, Modular forms of half-integral weight and the integral of certain theta functions, *Nagoya Math. J.* **56** (1974) 147–161
[7] Pizer A, Hecke operators for $\Gamma_0(N)$, *J. Algebra* **83** (1983) 39–64
[8] Shimura G, On modular forms of half-integral weight, *Ann. Math.* **97** (1973) 440–481
[9] Shintani T, On construction of holomorphic cusp forms of half-integral weight, *Nagoya Math. J.* **58** (1975) 83–126
[10] Skoruppa N-P, Zagier D, Jacobi forms and a certain space of modular forms, *Invent. Math.* **94** (1988) 113–146
[11] Skoruppa N-P, Zagier D, A trace formula for Jacobi forms. *J. Reine Angew Math.* **393** (1989) 168–198
[12] Ueda M, The decomposition of the spaces of cusp forms of half-integral weight and trace formula of Hecke operators, *J. Math. Kyoto Univ.* **28–3** (1988) 505–555

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On the integral modulus of continuity of Fourier series

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Abstract. We obtain an estimate for the integral modulus of continuity of order k of Fourier series with coefficients satisfying: $a_v \to 0$ and $\sum_{v=1}^{\infty} v^2|\Delta^2(a_v/v)| < \infty$.

Keywords. Fourier series; integral modulus of continuity

1. Introduction

Let F be a function of period 2π in $L_p(1 \leqslant p < \infty)$. Then the integral modulus of continuity of order k of F in L_p is defined by

$$\omega_p^k(h; F) = \sup_{0<|t|\leqslant h} \|\Delta_t^k F(x)\|_{L_p},$$

where

$$\Delta_t^k F(x) = \sum_{\alpha=0}^{k} (-1)^{k-\alpha}\binom{k}{\alpha} F(x+\alpha t)$$

and $\|\cdot\|_{L_p}$ denotes the norm in L_p.

Let $g(x)$ denote the sum of the sine series $\sum_{v=1}^{\infty} b_v \sin vx$. Throughout this paper, the letter A with or without subscripts denotes a constant having different values in different contexts and depending upon the subscripts.

Concerning integral modulus of continuity of order 1, Izumi and Izumi [2] proved the following theorem.

Theorem A. *If the sequence $\{b_v\}$ is quasi-convex, that is,*

$$\sum_{v=1}^{\infty} (v+1)|\Delta^2 b_v| < \infty,$$

where $\Delta^2 b_v = \Delta b_v - \Delta b_{v+1} = b_v - 2b_{v+1} + b_{v+2}$, then

$$\omega_1\left(\frac{1}{n}, g\right) \leqslant \frac{A}{n}\sum_{v=1}^{n} v^2|\Delta^2 b_{v-1}| + A\sum_{v=n+1}^{\infty} v\left(1+\log\frac{v}{n}\right)|\Delta^2 b_{v-1}|.$$

(This is a corrected version of the result as given in Telyakovskiĭ [4]).

The following theorem of Telyakovskiĭ [4] is an improvement on Theorem A.

Theorem B. *Let $\{b_v\}$ be a quasi-convex null sequence satisfying $\sum_{v=1}^{\infty}(|b_v|/v) < \infty$, then*

the integral modulus of continuity of g satisfies the relation

$$\omega_1\left(\frac{1}{n}, g\right) \leqslant \frac{A}{n} \sum_{v=1}^{n} v^2 |\Delta^2 b_{v-1}| + A \sum_{v=n+1}^{\infty} v |\Delta^2 b_{v-1}| + A \sum_{v=n}^{\infty} \frac{|b_v|}{v}.$$

Kano [3] proved the following result:

Theorem C. *If $\{b_v\}$ is a null sequence such that*

$$\sum_{v=1}^{\infty} v^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right| < \infty, \tag{1.1}$$

then

$$\sum_{v=1}^{\infty} b_v \sin vx \tag{1.2}$$

is a Fourier series, or equivalently, it represents an integrable function g.

2. Results

The aim of this paper is to obtain an estimate for $\omega_1^k(h; g)$ under the condition (1·1). We establish the following theorem.

Theorem. *Let $\{b_v\}$ be a null sequence satisfying* (1·1). *Then*

$$\omega_1^k\left(\frac{1}{n}, g\right) \leqslant \frac{A_k}{n^k} \sum_{v=1}^{n} (v+1)^{k+2} \left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left|\Delta^2\left(\frac{b_v}{v}\right)\right|.$$

The case $k = 1$ of our theorem is analogous to Theorems A and B.

The following lemma will be needed in the proof of our theorem.

Lemma. Let $0 < t \leqslant 1/n$ ($n = 1, 2, \ldots$) and let k be a natural number. If $K_v(x)$ denotes the Fejér kernel, then

$$\int_0^\pi |\Delta_{\pm t}^k K_v'(x)| \,\mathrm{d}x \leqslant \begin{cases} A_k t^k v^{k+1}, & v = 1, 2, \ldots, n \\ A_k v, & v = 1, 2, \ldots. \end{cases} \tag{2.1}$$

Proof of the lemma. We have

$$\int_0^\pi |\Delta_{\pm t}^k K_v'(x)| \,\mathrm{d}x \leqslant A_k t^k \left(\int_0^{(k+1)/v} + \int_{(k+1)/v}^\pi\right) \left|K_v^{(k+1)}(x \pm \theta_v t)\right| \mathrm{d}x$$
$$= A_k t^k (J_1 + J_2),$$

with $0 < \theta_v < k$. Now because of

$$|K_v^{(k+1)}(x)| \leqslant \begin{cases} A_k v^{k+2}, & 0 \leqslant x \leqslant \pi \\ A_k v^k x^{-2}, & 0 < x \leqslant \pi \end{cases} \quad (k = 1, 2, \ldots),$$

we have $J_1 \leqslant A_k v^{k+1}$ and

$$J_2 \leqslant A_k v^k \int_{(k+1)/v}^{\pi} \frac{1}{(x \pm \theta_v t)^2} \mathrm{d}x \leqslant A_k v^k \int_{(k+1)/v}^{\pi} \frac{\mathrm{d}x}{(x-kt)^2}.$$

By virtue of $t \leqslant 1/n$ and $v = 1, 2, \ldots, n$, $x \geqslant (k+1)/v$; we have $x \geqslant (k+1)/n \geqslant (k+1)t$. This yields

$$\frac{1}{x-kt} \leqslant \frac{k+1}{x} (x \geqslant (k+1)t).$$

Thus, we obtain

$$J_2 \leqslant A_k v^k \int_{(k+1)v}^{\pi} \frac{\mathrm{d}x}{x^2} \leqslant A_k v^k \int_{(k+1)/v}^{\infty} \frac{\mathrm{d}x}{x^2} \leqslant A_k v^{k+1}.$$

This proves first part of the inequality (2.1). To prove the second part of (2.1), we use Zygmund's Theorem [1, p. 458] and have

$$\begin{aligned}
\int_0^{\pi} |\Delta_{\pm t}^k K_v'(x)| \,\mathrm{d}x &\leqslant \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_0^{\pi} |K_v'(x+\alpha t)| \,\mathrm{d}x \\
&\leqslant \sum_{\alpha=0}^{k} \binom{k}{\alpha} \int_{-\pi}^{\pi} |K_v'(x)| \,\mathrm{d}x \\
&\leqslant A_k v \int_{-\pi}^{\pi} |K_v(x)| \,\mathrm{d}x \\
&= \pi A_k v.
\end{aligned}$$

Proof of the theorem. Theorem C implies that g is integrable. Let

$$S_n(x) = \sum_{v=1}^{n} b_v \sin vx.$$

Two applications of Abel's transformation yield

$$\begin{aligned}
S_n(x) &= -\frac{\mathrm{d}}{\mathrm{d}x} \sum_{v=1}^{n} \frac{b_v}{v} \cos vx \\
&= -\left[\sum_{v=1}^{n-1} \Delta\left(\frac{b_v}{v}\right)(D_v'(x) - \tfrac{1}{2}) + \frac{b_n}{n}(D_n'(x) - \tfrac{1}{2})\right] \\
&= -\left[\sum_{v=1}^{n-2} (v+1)\Delta^2\left(\frac{b_v}{v}\right) K_v'(x) + n\Delta\left(\frac{b_{n-1}}{n-1}\right) K_{n-1}'(x)\right] \\
&\quad + \frac{1}{2}\sum_{v=1}^{n-1} \Delta\left(\frac{b_v}{v}\right) - \frac{b_n}{n} D_n'(x) + \frac{1}{2}\frac{b_n}{n},
\end{aligned}$$

where $D_v(x)$ and $K_v(x)$ denote Dirichlet kernel and Fejér kernel respectively. Then [3, p. 159]

$$g(x) = \lim_{n\to\infty} S_n(x) = -\sum_{v=1}^{\infty} (v+1)\Delta^2\left(\frac{b_v}{v}\right) K_v'(x).$$

Since the symmetry of the function g implies

$$|\Delta_t^k g(-x)| = |\Delta_{-t}^k g(x)|,$$

therefore

$$\int_{-\pi}^{\pi} |\Delta_t^k g(x)|\, \mathrm{d}x = \int_0^{\pi} |\Delta_{-t}^k g(x)|\, \mathrm{d}x + \int_0^{\pi} |\Delta_t^k g(x)|\, \mathrm{d}x.$$

Hence, to prove our theorem, it is sufficient to estimate

$$\int_0^{\pi} |\Delta_{\pm t}^k g(x)|\, \mathrm{d}x \quad \text{for } 0 < t \leqslant 1/n.$$

We write

$$\begin{aligned}\int_0^{\pi} |\Delta_{\pm t}^k g(x)|\, \mathrm{d}x &= \int_0^{\pi} \left|\Delta_{\pm t}^k \sum_{v=1}^{\infty} (v+1)\Delta^2\left(\frac{b_v}{v}\right) K_v'(x)\right| \mathrm{d}x \\ &= \int_0^{(k+1)/n} + \int_{(k+1)/n}^{\pi} \\ &= I_1 + I_2.\end{aligned}$$

We first estimate I_2. We have

$$\begin{aligned} I_2 &\leqslant \left(\sum_{v=1}^{n} + \sum_{v=n+1}^{\infty}\right)\left[(v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right| \int_{(k+1)/n}^{\pi} |\Delta_{\pm t}^k K_v'(x)|\, \mathrm{d}x\right] \\ &= I_{21} + I_{22}, \text{ say.}\end{aligned}$$

Now, by first part of the inequality (2.1), we have

$$\begin{aligned} I_{21} &= \sum_{v=1}^{n} (v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right| \int_{(k+1)/n}^{\pi} |\Delta_{\pm t}^k K_v'(x)|\, \mathrm{d}x \\ &\leqslant A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2}\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.\end{aligned}$$

The second part of the inequality (2.1) implies

$$\begin{aligned} I_{22} &= \sum_{v=n+1}^{\infty} (v+1)\left|\Delta^2\left(\frac{b_v}{v}\right)\right| \int_{(k+1)/n}^{\pi} |\Delta_{\pm t}^k K_v'(x)|\, \mathrm{d}x \\ &\leqslant A_k \sum_{v=n+1}^{\infty} (v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|.\end{aligned}$$

Thus

$$I_2 \leqslant A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2}\left|\Delta^2\left(\frac{b_v}{v}\right)\right| + A_k \sum_{v=n+1}^{\infty} (v+1)^2\left|\Delta^2\left(\frac{b_v}{v}\right)\right|. \tag{2.2}$$

To estimate I_1, we have

$$I_1 = \int_0^{(k+1)/n} \left| \Delta^k_{\pm t} \left(\sum_{v=1}^{\infty} (v+1) \Delta^2 \left(\frac{b_v}{v} \right) K'_v(x) \right) \right| \mathrm{d}x$$
$$\leqslant \left(\sum_{v=1}^{n} + \sum_{v=n+1}^{\infty} \right) \left[(v+1) \left| \Delta^2 \left(\frac{b_v}{v} \right) \right| \int_0^{(k+1)/n} |\Delta^k_{\pm t} K'_v(x)| \, \mathrm{d}x \right]$$
$$= I_{11} + I_{12}, \text{ say.}$$

Now, using first part of (2.1), we have

$$I_{11} = \sum_{v=1}^{n} (v+1) \left| \Delta^2 \left(\frac{b_v}{v} \right) \right| \int_0^{(k+1)/n} |\Delta^k_{\pm t} K'_v(x)| \, \mathrm{d}x$$
$$\leqslant A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2} \left| \Delta^2 \left(\frac{b_v}{v} \right) \right|$$

and using the second part of (2.1), we have

$$I_{12} = \sum_{v=n+1}^{\infty} (v+1) \left| \Delta^2 \left(\frac{b_v}{v} \right) \right| \int_0^{(k+1)/n} |\Delta^k_{\pm t} K'_v(x)| \, \mathrm{d}x$$
$$\leqslant A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left| \Delta^2 \left(\frac{b_v}{v} \right) \right|.$$

We have therefore

$$I_1 \leqslant A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2} \left| \Delta^2 \left(\frac{b_v}{v} \right) \right| + A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left| \Delta^2 \left(\frac{b_v}{v} \right) \right| \tag{2.3}$$

Combining (2.2) and (2.3), it follows that

$$\int_0^{\pi} |\Delta^k_{\pm t} g(x)| \, \mathrm{d}x \leqslant A_k n^{-k} \sum_{v=1}^{n} (v+1)^{k+2} \left| \Delta^2 \left(\frac{b_v}{v} \right) \right|$$
$$+ A_k \sum_{v=n+1}^{\infty} (v+1)^2 \left| \Delta^2 \left(\frac{b_v}{v} \right) \right|.$$

This completes the proof of the theorem.

References

[1] Bari N K, "A treatise on trigonometric series", Vol. II, (New York: Pergamon Press) (1964)
[2] Izumi M and Izumi S, Modulus of continuity of functions defined by trigonometric series, *J. Math. Anal. Appl.* **24** (1968), 564–581
[3] Kano T, Coefficients of some trigonometric series, *J. Fac. Sci. Shinshu Univ.* **3** (1968), 153–162
[4] Telyakovskiĭ S A, The integral modulus of continuity of functions with quasi-convex Fourier coefficients, *Sib. Mat. Zh.* **11** (1970), 1140–1145

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On orbit equivalence of Borel automorphisms

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Abstract. Let E and F be two Borel sets of the countable product Z of the two point space $\{0, 1\}$. Assume that E and F are invariant sets for the odometer transformation R and that E and F are of measure zero with respect to the unique finite R-invariant measure on Z. We show that E and F are R-orbit equivalent in a strict sense.

Keywords. Odometer transformation; orbit equivalence of Borel automorphisms; compressibility.

1. Introduction

Let $Z = \{0, 1\}^{\aleph_0}$ be the countably infinite product of the two point space $\{0, 1\}$. We equip Z with the product σ-algebra (denoted by $\mathscr{A}$) and the product probability measure (denoted by P), where the two point space $\{0, 1\}$ is given the discrete σ-algebra and the uniform probability measure $p : p\{0\} = p\{1\} = 1/2$. The odometer transformation R on Z is defined as follows: If $z = (z_1, z_2, z_3 \ldots)$ be a point εZ and if n be the first positive integer for which $z_n = 0$, then the image $\omega = Rz$ of z is given by

$$\omega_k = \begin{cases} 0 & \text{if} \quad k < n \\ 1 & \text{if} \quad k = n \\ z_k & \text{if} \quad k > n \end{cases}$$

where ω_k denotes the kth co-ordinate of ω. If $z = (1, 1, 1, \ldots)$, then $Rz = (0, 0, 0, \ldots)$. It is known that R is uniquely ergodic, the measure P being the unique R-invariant probability measure on Z. Let E and F be two R invariant Borel subsets of Z each of P-measure zero. Let U and V denote the restrictions of R to E and F respectively. The purpose of this paper is to prove theorem 1.

Theorem 1. *If the orbit spaces of U and V do not admit Borel cross-sections, then U and V are orbit equivalent, i.e., there exists a Borel isomorphism φ of E onto F such that for each $x \in E$,*

$$\{V^n \varphi(x)\}_{n=-\infty}^{\infty} = \varphi(\{U^n x\}_{n=-\infty}^{\infty}).$$

This result may be viewed as a contribution to the theme of descriptive ergodic theory. (See [1], [6], [7], [9], [10], [11]). The main contribution here is lemma 1

below, which when coupled with some known facts and Cantor-Bernstein theorem for orbit equivalence, yields theorem 1 above.

2. Compressibility of null invariant sets in Z

Let $(X,\mathscr{B})$ be a standard Borel space and T a Borel automorphism on X. For any $A \subseteq X$, we write sA to denote the T-invariant set $\bigcup_{n=-\infty}^{\infty} T^n A$ generated by A, and we call sA the saturation of A. Two sets $A, B \in \mathscr{B}$ are said to be equivalent by countable decomposition if (i) we can partition A into a countable number of pairwise disjoint sets $A_i \in \mathscr{B}$, $i \in \mathbb{N}$ (ii) we can partition B into a countable number of pairwise disjoint sets $B_i \in \mathscr{B}$, $i \in \mathbb{N}$ (iii) we can find integers n_i, $i \in \mathbb{N}$ such that for each i, $T^{n_i} A_i = B_i$. Here and in the sequel $\mathbb{N}$ will denote the set of natural numbers. We write $A \sim B$ whenever A and B are equivalent by countable decomposition. If $A \sim B$, then the map $\varphi: A \to B$ given by $\varphi = T^{n_i}$ on A_i, (where A_i, B_i, n_i are as above) is called a descriptive isomorphism between A and B. If $A \sim B$ we also say that A and B are descriptively isomorphic.

DEFINITION

We say that a set $A \in \mathscr{B}$ is compressible if A can be expressed as a disjoint union of two sets B and C in $\mathscr{B}$ such that

(i) $sA = sB = sC$, and (ii) $A \sim B$.

If m is a countably additive T-invariant measure on $\mathscr{B}$, then two descriptively isomorphic sets in $\mathscr{B}$ have the same m-measure. Further a compressible set of finite m-measure has necessarily zero m-measure. If T is uniquely ergodic and m is the unique T-invariant probability measure on $\mathscr{B}$, then any compressible T-invariant set in $\mathscr{B}$ has m-measure zero. It seems natural to conjecture that if T is uniquely ergodic then any T-invariant set in $\mathscr{B}$ of m-measure zero is compressible. We will verify this conjecture for odometer transformation.

DEFINITION

Given a standard Borel space $(X,\mathscr{B})$ and a Borel automorphism T on X, we say that T is set periodic with period k if there is a partition

$$\mathscr{D}_k = \{D_1, D_2, \ldots, D_k\}$$

of X associated with T such that

$$D_i = T^{i-1} D_1, \quad 1 \leqslant i \leqslant k.$$

If for each $n \in \mathbb{N}$, T is set periodic with period 2^n and with associated partition $\mathscr{D}_n(T) = \{D_1^n, \ldots, D_{2^n}^n\}$, such that $D_i^n = D_i^{n+1} \cup D_{i+2^n}^{n+1}$, $i = 1, 2, \ldots, 2^n$, $n \in \mathbb{N}$, then we call T a weak von-Neumann transformation. We call T a von-Neumann transformation if T is a weak von-Neumann transformation and the union $\bigcup_{n=1}^{\infty} \mathscr{D}_n(T)$ of the associated sequence of partitions generates the σ-algebra $\mathscr{B}$.

The odometer transformation R on Z and the restrictions of R to R-invariant Borel sets are all von-Neumann transformations. Conversely any von-Neumann transforma-

tion is isomorphic to the restriction of R to a suitable R-invariant Borel subset of Z. A von-Neumann transformation V on $(X, \mathscr{B})$ admits utmost one V-invariant countably additive probability measure on $\mathscr{B}$.

Lemma 1. *If a von-Neumann transformation V on a standard Borel space $(X, \mathscr{B})$ does not admit a V-invariant countably additive probability measure on $\mathscr{B}$, then X is compressible (with respect to V).*

Proof. Let $\mathscr{D}_n = \{D_1^n, \ldots, D_{2^n}^n\}$ be the sequence of partitions associated to V as per the definition of von-Neumann transformation. Let $\mathscr{P}_n$ denote the algebra generated by $\mathscr{D}_n$. We have $\mathscr{P}_n \subseteq \mathscr{P}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathscr{P}_n = \mathscr{P}$ is again an algebra. On $\mathscr{P}$ we define a V-invariant finitely additive measure m by setting, for all n, $m(D_k^n) = 1/2^n$, $1 \leqslant k \leqslant 2^n$. We will need the following two observations:

(i) If $A, B \in \mathscr{P}_n$ and $m(A) \leqslant m(B)$ then there is a set $C \in \mathscr{P}_n$, $C \subseteq B$, such that $A \sim C$, $m(A) = m(C)$ and $m(B - C) = m(B) - m(A)$. The sets A and C are in fact equivalent by finite decomposition through sets in $\mathscr{P}_n$.
(ii) If $A, B \in \mathscr{P}$ and $m(A) \leqslant m(B)$, then there is a set $C \subseteq B$, $C \in \mathscr{P}$, such that $A \sim C$, $m(A) = m(C)$ and $m(B - C) = m(B) - m(A)$. The sets A and C are in fact equivalent by finite decomposition by sets in $\mathscr{P}$. This follows (i) because for large enough n, $A, B \in \mathscr{P}_n$.

Since there is no V-invariant countably additive probability measure on $\mathscr{B}$, the finitely additive measure m on $\mathscr{P}$ is not countably additive on $\mathscr{P}$. (For if m were countably additive on $\mathscr{P}$, it would extend to a V-invariant countably additive probability measure on the σ-algebra generated by $\mathscr{P}$, which is $\mathscr{B}$). Therefore there exist pairwise disjoint sets $\delta_1, \delta_2, \delta_3, \ldots$ in $\mathscr{P}$ such that $X = \bigcup_{n=1}^{\infty} \delta_n$ and $\sum_{n=1}^{\infty} m(\delta_n) \lneqq 1$. There is no loss of generality if we assume that $\delta_1 \in \mathscr{P}_1$, $\delta_2 \in \mathscr{P}_2, \ldots, \delta_n \in \mathscr{P}_n, \ldots$. (Some of the δ_n's could be empty). Let $q = 1 - \sum_{n=1}^{\infty} m(\delta_n) > 0$ and choose a positive integer N such that $2^{-N} < \frac{1}{2}q$. Recalling that $\mathscr{D}_N = \{D_1^N, \ldots, D_{2^N}^N\}$ denotes the Nth partition associated to V, we set $B = X - D_{2^N}^N$ and $C = D_{2^N}^N$. The sets B, C belong to $\mathscr{P}_N$, hence to $\mathscr{P}$ and $\mathscr{B}$. We note that $sC = sB = sX = X$. We now show that $X \sim B_1 \subseteq B$, thus proving the compressibility of X. We have $m(\delta_1) \leqslant 1 - q < 1 - 1/2N = m(B)$. Hence there exists, by observation (ii), a $r_1 \subseteq B$ such that $r_1 \in \mathscr{P}$, $\delta_1 \sim r_1$ and

$$m(B - r_1) = m(B) - m(\delta_1) = 1 - 1/2^N - m(\delta_1) > 1 - q - m(\delta_1)$$
$$= \sum_{i=2}^{\infty} m(\delta_i) \geqslant m(\delta_2).$$

Again by observation (ii) there exists $r_2 \subseteq B - r_1$, $r_2 \in \mathscr{P}$, such that $\delta \sim r_2$ and

$$\begin{aligned} m(B - r_1 - r_2) &= m(B - r_1) - m(\delta_2) = m(B) - m(\delta_1) - m(\delta_2) \\ &= 1 - 1/2^N - m(\delta_1) - m(\delta_2) > 1 - q - m(\delta_1) - m(\delta_2) \\ &= \sum_{i=3}^{\infty} m(\delta_i) \geqslant m(\delta_3). \end{aligned}$$

Proceeding thus we can find $r_1, r_2, r_3, \ldots$, inside B such that for each i, $r_i \in \mathscr{P}$, $r_i \sim \delta_i$. Thus X is equivalent by countable decomposition to $\bigcup_{i=1}^{\infty} r_i \subseteq B$. Since $B \subseteq X$ we see that $X \sim B$. (See [7] 5.3). q.e.d.

Remark. The sets E and F of theorem 1 are compressible with respect to U and V respectively since U and V are von-Neumann transformations which do not admit countably additive invariant probability measures in view of unique ergodicity of R.

3. A Cantor-Bernstein theorem

We need Cantor-Bernstein theorem for orbit equivalence which is as follows.

Theorem 2. *Let $(X,\mathscr{B})$, $(Y,\mathscr{C})$ be standard Borel spaces. Let S and T be Borel automorphisms on X and Y respectively such that T is orbit equivalent to the restriction of S to an S invariant Borel set $A \subseteq X$ and S is orbit equivalent to the restriction of T to a T invariant Borel set $B \subseteq Y$. Then S and T are orbit equivalent.*

Proof. Let $f: X \to B \subseteq Y$ be a one-one onto Borel map such that for all $x \in X$

$$f(\{S^n x\}_{n=-\infty}^{\infty}) = \{T^n f(x)\}_{n=-\infty}^{\infty}.$$

Similarly, let $g: Y \to A \subseteq X$ be a one-one onto Borel map such that for all $y \in Y$,

$$g(\{T^n y\}_{n=-\infty}^{\infty}) = \{S^n g(y)\}_{n=-\infty}^{\infty}.$$

We now adapt one of the proofs of Cantor-Bernstein theorem. We have for $n \geqslant 0$

$$(g\circ f)^{n-1}g(Y) \supseteq (g\circ f)^n(X) \supseteq (g\circ f)^n g(Y).$$

The sets

$$X - g(Y),\ g(Y) - (g\circ f)(X), \ldots, (g\circ f)^{n-1}g(Y) - (g\circ f)^n(X),$$

$$(g\circ f)^n(X) - (g\circ f)^n g(Y), \ldots, \text{together with}$$

$$\bigcap_{n=0}^{\infty} (g\circ f)^n(X), \text{ form a countable partition of } X$$

into S-invariant Borel sets. We define $h: X \to Y$ as follows:

$$h = \begin{cases} f & \text{on } (g\circ f)^{n-1}g(Y) - (g\circ f)^n(X) \\ g^{-1} & \text{on } (g\circ f)^n(X) - (g\circ f)^n g(Y) \\ f & \text{on } \bigcap_{n=0}^{\infty} (g\circ f)^n(X) \end{cases}$$

where $n \geqslant 0$. The map h is clearly one-one, Borel and takes an S orbit onto a T-orbit. It remains to show that h is onto. For $n \geqslant 0$,

$$h((g\circ f)^{n-1}g(Y) - (g\circ f)^n(X)) = (f\circ g)^n(Y) - (f\circ g)^n f(X) \tag{1}$$

$$h((g\circ f)^n(X) - (g\circ f)^n g(Y)) = (f\circ g)^{n-1}f(X) - (f\circ g)^n(Y) \tag{2}$$

$$h\left(\bigcap_{n=0}^{\infty} (g\circ f)^n(X)\right) = \bigcap_{n=0}^{\infty} (f\circ g)^n f(X) = \bigcap_{n=0}^{\infty} (f\circ g)^n(Y) \tag{3}$$

The sets on the right hand side of (1) and (2) for all $n \geqslant 0$ together with the set on the right hand side of (3) give a partition of Y. This proves that h is onto, and completes the proof of the theorem.

4. Equivalence of tower with the base

Let T be a Borel automorphism on a standard Borel space $(X, \mathscr{B})$ and assume that T is free, i.e., T has no periodic points. A set W in $\mathscr{B}$ is said to be wandering if $T^n W \cap T^m W = \varnothing$ whenever $m \neq n$. The σ-ideal generated by wandering sets is denoted by $\mathbb{W}$ and called the Shelah-Weiss ideal (see [7]). If $A \in \mathscr{B}$, and, A_0 be the set of all those points $x \in A$ such that $T^n x$ return to A for infinitely many positive values of n and also for infinitely many negative values of n, then $A - A_0$ belongs to $\mathbb{W}$. If $A = A_0$ and A is compressible then the sets B and C needed in the definition of compressibility can be so chosen that $B = B_0$ and $C = C_0$.

Lemma 2. *If $A = A_0$ and A is compressible then $sA \sim A$.*

Proof. Since A is compressible and $A = A_0$, we can write $A = B \cup C$, $B \cap C = \varnothing$, $B = B_0$, $C = C_0$ where further $sA = sB = sC$ and $A \sim B$. Let $S: A \to B$ be a descriptive isomorphism. Then $C = A - B = A - SA$. The sets C, SC, $S^2C, \ldots$ are all pairwise disjoint and contained in A. Further, since

$$C = C_0, \; sC = \bigcup_{n=0}^{\infty} T^n C.$$

Put $C_1 = TC_0 - C_0$, and inductively, $C_j = TC_{j-1} - C_0, j \in \mathbb{N}$. The sets $C_j, j = 0, 1, 2, \ldots$, are pairwise disjoint and $\bigcup_{j=0}^{\infty} C_j = \bigcup_{n=0}^{\infty} T^n C_0 = sA$. Indeed C_j's are the levels of the Kakutani sky scraper construction with base $C = C_0$. (See [2]). We now define the map $S^*: sC_0 \to B = A - C$ as follows:

$$S^* x = S^{j+1} T^{-j} x, \quad x \in C_j, \quad j = 0, 1, 2, 3, \ldots.$$

Then

$$S^* C_j \subseteq S^{j+1} C_0. \text{ Since } C_j, j = 0, 1, 2, \ldots$$

are pairwise disjoint and make up sA, and, since $S^{j+1} C_0, j = 0, 1, 2, \ldots$, are pairwise disjoint and contained in A, we have sA descriptively isomorphic to a subset of A. Since $A \subseteq sA$, we have $A \sim sA$. (See [7] Lemma 5.3 and [3] corollary 1.7).

If $A = A_0$ one can define a transformation T_A on A, the so-called induced transformation, by $T_A x = T^{n(x)} x$, where $n(x)$ is the first positive integer such that $T^{n(x)} x \in A$. Of course, x, to begin with is in A. If, for any two sets $A, B \in \mathscr{B}$, $A = A_0$, $B = B_0$ and A and B are descriptively isomorphic, then T_A and T_B are orbit equivalent, the transformation S which establishes the descriptive isomorphism between A and B also establishes the orbit equivalence between T_A and T_B. We see therefore, in view of lemma 2, that if $A = A_0$ and A is compressible then T_A and the restriction of T to sA are orbit equivalent.

5. Proof of theorem 1

We are now in a position to prove theorem 1. A theorem of Glimm and Effros coupled with a result of Ramsay and Mackey (see [10], [7]) permits one to conclude that if the orbit space of a Borel automorphism T on a standard Borel space $(X, \mathscr{B})$ does not admit a Borel cross section, then there exists a Borel set $A \subseteq X$ such that T_A is

orbit equivalent to the odometer transformation R on $\{0,1\}^{N}$. Passing to a subset we conclude that if the orbit space of T does not admit a Borel cross section then there exists a set $A \subseteq X$, A Borel, such that T_A is orbit equivalent to U of theorem 1. Applying this fact with $T = V$ and $X = F$, we see that there is a Borel subset $F_1 \subseteq F$ such that V_{F_1} is orbit equivalent to U. Since E is compressible by lemma 1, we conclude that F_1 is compressible. Since F_1 is compressible, lemma 2 shows that V_{F_1} is orbit equivalent to the restriction of V to sF_1. Thus U is orbit equivalent to the restriction of V to a V invariant Borel subset of F. Similarly V is orbit equivalent to the restriction of U to a U-invariant Borel subset of E. The Cantor-Bernstein theorem of §3 yields now theorem 1.

Remarks. If m is an atom free probability measure on a standard Borel space $(X, \mathscr{B})$ and T a Borel automorphism on X which preserves m-null sets and is ergodic with respect to m, then it is known (see [8]) that the restriction of T to a suitable T-invariant Borel set (say Y) of full m-measure is orbit equivalent to the restriction of the odometer R to an R-invariant Borel set, say E. Further if there is no finite T-invariant measure on $\mathscr{B}$ with same null sets as m, then the R-invariant set E has P-measure zero. Since m is atom free and T is ergodic with respect to m, the orbit space of T (restricted to Y) does not admit a Borel cross section, a fortiriori, the orbit space of R restricted to E does not admit a Borel cross section. These facts together with theorem 1 permit us to conclude that if S and T are two non-singular ergodic automorphisms on $(X, \mathscr{B}, m)$ neither admitting finite invariant measure with same null sets as m, then there exists an S-invariant Borel set Y_1 of full m-measure and a T invariant Borel set Y of full m-measure such that $S|_{Y_1}$ and $T|_Y$ are orbit equivalent. This result does not contradict Krieger's work ([4], [5]) on weak equivalence via ratio sets because the Borel isomorphism between Y_1 and Y which implements the orbit equivalence is not claimed to preserve the m-null sets.

It seems natural to conjecture that any two compressible and free Borel automorphisms S and T (i.e. X is compressible with respect to both S and T) whose orbit spaces do not admit Borel cross sections are orbit equivalent. In [7] it is proved that any such Borel automorphism is orbit equivalent to a weak von-Neumann transformation. If the word "weak" in this result can be removed, the conjecture would follow from theorem 1. It seems natural to conjecture, also, that any two free Borel automorphism are orbit equivalent if and only if the cardinality of the ergodic invariant probability measures they admit is the same. In particular it seems that any two uniquely ergodic free Borel automorphism on a standard Borel space are orbit equivalent without discarding any null sets. Papers [1] and [7] are relevant for some of these questions, but it should be recorded here that there is a gap in the proof of theorem 2 of paper [1] and that a correct proof is available under the additional condition (*) stated in §3 of the same paper.

References

[1] Chaube P and Nadkarni M G, A version of Dye's theorem for descriptive dynamical systems, *Sankhya, The Indian J. Stat.* **49** (1987) Series A pt 3 pp. 288–304. See also a correction to appear in the same journal

[2] Friedman N, *Introduction to Ergodic Theory* (Reinhold, New Jersey: Van Nostrand) (1970)

[3] Katznelson Y and Weiss B, Notes on Orbit Equivalence, Draft

4] Krieger W, *On non-singular transformations of a measure space I, II, Z. Wahrcheinlichkeitstheorie verw. Geb.* 11 (1969) pp. 83–97, 98–119
5] Krieger W, Araki-Woods asymptotic ratio sets and non-singular transformation of a measure space, Contributions to Ergodic Theory and Probability, Springer Lecture Notes in Mathematics No. 160 (1970) pp. 158–177
6] Nadkarni M G, A descriptive characterization of Ergodic Systems, *Sankhya* **45** (1983) pt. 3 pp. 271–286
7] Nadkarni M G, Descriptive Ergodic Theory, Contemperory Mathematics, *Stone Proceedings*, (Providence, R.I: American Mathematical Society)
8] Schmidt K, Cocycles on Ergodic Transformation Groups, MacMillan Lecture Notes in Mathematics (1977) N. Delhi
9] Wagh V, A descriptive version of Ambrose' representation theorem for flows, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **98** (1988) pp. 101–108
0] Weiss B, Measurable Dynamics, *Contemperory Mathematics*, 1984, Amer. Math. Soc. **26** pp. 295–421
1] Weiss B, Countable generators in Dynamics Universal minimal Models, preprint

Proc. Indian Acad. Sci. (Math. Sci.), Vol. 99, No. 3, December 1989, pp. 263–272.

Steady-state thermoelastic stresses in an infinite transversely isotropic medium containing an external circular crack

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Abstract. The temperature and the normal components of stress and displacement around an external circular crack in an infinite transversely isotropic body have been calculated in the present paper. The stress intensity factor has been found and a comparison of the results with those for the isotropic case has been presented graphically.

Keywords. Infinite transversely isotropic body; stress intensity factor.

1. Introduction

Elliot [1] solved the problem of stress distribution in hexagonal aerolotropic crystals in the absence of body forces in terms of two stress functions each satisfying a second order partial differential equation. Singh [2] extended the method to solve the thermoelastic punch problems when body forces are present. It appears that thermoelastic mixed boundary value problems for anisotropic media containing cracks have not been considered appreciably. The present paper employs Singh's method and deals with the determination of the temperature and the components of stress and displacement in a transversely isotropic solid containing a flat external crack covering the outside of a circle. The faces of the crack are exposed to a prescribed axisymmetric temperature. The stress intensity factor is found and the normal components of stress and displacement, together with those obtained in the corresponding isotropic problem by Das [3], are presented graphically.

2. Notations

The following notations are used in this paper

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

$$a = (S_{11}S_{33} - S_{13}^2)/{}_1S_2S_{11}$$

$$c = (\beta S_{11} - \alpha S_{13})/{}_1S_2S_{11}$$

$$a_{11} = CK_{21}^2 - \frac{\alpha}{S_{11}}$$

K_1^2, K_2^2 constants depending upon elastic constants. A_1, A_2, A_3 integration constants.

$$h_1 = a_{11}L^2/p^2(K_{21}^2 - L^2) = \frac{M}{p^2}$$

$$h_2 = (K_{12}^2 - K_{11}^2)^{-1}$$

$$h_3 = L^2\left(CL^2 - \frac{\alpha}{S_{11}}\right)\Big/\Big[p^2 a(K_{22}^2 - L^2)(K_{21}^2 - L^2) = \frac{N}{p^2}$$

$$(n, n_1, n_2) = \left(\frac{p}{L}, \frac{p}{K_{21}}, \frac{p}{K_{22}}\right)$$

3. Basic equations

We assume that the infinite thermoelastic medium contains a flat external crack covering the exterior of a circle of unit radius. With origin of co-ordinates at the centre of this circle and the axis of z perpendicular to the plane of the circle, the position of a typical point of the solid may conveniently be expressed by the cylindrical polar co-ordinates (r, θ, z). We further assume that the axis of elastic symmetry is coincident with the z axis and that the thermal properties in the directions of r and θ increasing are same but different in the z-direction. In axially symmetric deformations the displacement vector will have the components $(u_r, 0, u_z)$. The non-vanishing components of the stress and strain tensors will be σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, σ_{rz} and e_{rr}, $e_{\theta\theta}$, e_{zz}, e_{rz} respectively.

The strain-stress relations for the transversely isotropic material are assumed in the form

$$\begin{bmatrix} e_{rr} \\ e_{\theta\theta} \\ e_{zz} \\ e_{rz} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 \\ S_{12} & S_{11} & S_{13} & 0 \\ S_{13} & S_{13} & S_{33} & 0 \\ 0 & 0 & 0 & S_{44} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{rz} \end{bmatrix} \tag{3.1}$$

where S_{11}, S_{12}, S_{13}, S_{33}, S_{44} are elastic constants.

In his paper, Singh [2] proves that the solutions of thermoelastic problems for a semi-infinite medium can be found in terms of two functions $\Phi(r, z)$ and $\Omega(r, z)$ which, in the absence of body forces, satisfy the differential equations

$$\nabla_1^2\Phi + K_{22}^2\frac{\partial^2\Phi}{\partial z^2} = \frac{1}{a}\left(\frac{\partial^2\Omega}{\partial z^2} - CT\right) \tag{3.2}$$

and

$$\nabla_1^2\Omega + K_{21}^2\frac{\partial^2\Omega}{\partial z^2} = a_{11}T \tag{3.3}$$

where K_{21}^2 and K_{22}^2 are the roots of the equation

$$(S_{13}^2 - S_{11}S_{33})K_2^4 - [2S_{13}(S_{12} - S_{11}) - S_{11}S_{44}]K_2^2 + (S_{12}^2 - S_{11}^2) = 0 \tag{3.4}$$

$T(r, z)$ denotes temperature and α, β denote the co-efficients of linear thermal expansion

along z-axis and in z-plane. The components of the stress tensor and the displacement vector are then given by

$$\dot{\sigma}_{rr}=\frac{\partial^2\phi}{\partial z^2}+\frac{1}{r}\left(K_{11}^2\frac{\partial^2\phi}{\partial r}+\frac{\partial\Omega}{\partial r}\right) \tag{3.5}$$

$$\sigma_{\theta\theta}=-\frac{1}{S_{11}}\left[S_{13}\nabla_1^2\phi+S_{12}\frac{\partial^2\phi}{\partial z^2}+\frac{S_{11}}{r}\left(K_{11}^2\frac{\partial\phi}{\partial r}+\frac{\partial\Omega}{\partial r}+\alpha T\right)\right] \tag{3.6}$$

$$\sigma_{zz}=\nabla_1^2\phi \tag{3.7}$$

$$\sigma_{rz}=-\frac{\partial^2\phi}{\partial r\partial z} \tag{3.8}$$

$$u_r=-{}_1S_2\left(K_{11}^2\frac{\partial\phi}{\partial r}+\frac{\partial\Omega}{\partial r}\right) \tag{3.9}$$

$$u_z=-{}_1S_2\left(K_{12}^2\frac{\partial\phi}{\partial z}-\frac{\partial\Omega}{\partial z}\right). \tag{3.10}$$

where K_{11}^2 and K_{12}^2 are the roots of the equation

$$S_{11}(S_{12}-S_{11})K_1^4+S_{11}S_{44}K_1^2+[S_{13}(2S_{13}+S_{44})-S_{33}(S_{11}+S_{12})]=0. \tag{3.11}$$

The equation of steady state heat conduction is

$$\nabla_1^2T+L^2\frac{\partial^2T}{\partial z^2}=0 \tag{3.12}$$

where L^2 is the ratio of the co-efficients of thermal conductivities along z axis and in z-plane.

4. Boundary conditions

With a suitable choice of unit of length we can assume that the faces of the crack are described by the relations $z=0\pm$, $r\geqslant 1$. We suppose that there is no external force acting on the crack faces and that the face $z=0+$, $r\geqslant 1$ is heated (or cooled) exactly in the same way as the face $z=0-$, $r\geqslant 1$. Then following Sneddon [4] we reduce the crack problem for the infinite medium $|z|\geqslant 0$ to the mixed boundary value problem for the semi-infinite medium $z\geqslant 0$ for which the thermal and elastic conditions on the boundary $z=0$ are

$$\frac{\partial T}{\partial z}(r,0)=0,\quad 0\leqslant r<1 \tag{4.1}$$

$$T(r,0)=f(r),\quad 1<r<\infty \tag{4.2}$$

$$\sigma_{r,z}(r,0)=0,\quad 0\leqslant r<\infty \tag{4.3}$$

$$\sigma_{zz}(r,0)=0,\quad 1<r<\infty \tag{4.4}$$

$$u_z(r,0)=0,\quad 0\leqslant r<1. \tag{4.5}$$

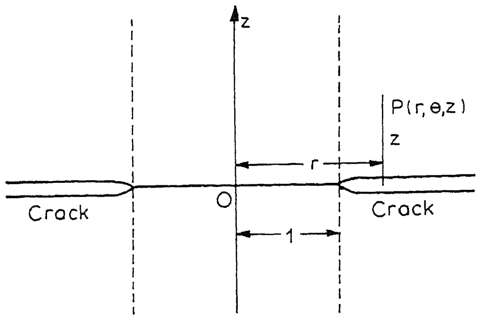

Figure 1.

We further assume that the disturbance is localized i.e. the temperature and the components of stress and displacement all vanish as $(r^2+z^2)^{1/2} \to \infty$.

An axial section of the medium through the field point $P(r,\theta,z)$ is shown in figure 1. This gives an idea about the overall configuration and the system of cylindrical polar coordinates used in the problem.

5. The heat conduction problem

Using the method of Singh [2], a suitable solution of the steady state heat conduction equation (3.12) vanishing at infinity is taken in the form:

$$T(r,z)=\int_0^\infty p\exp(-nz)A_3(p)J_0(rp)\,\mathrm{d}p \tag{5.1}$$

where $A_3(p)$ is an unknown function to be determined from the thermal conditions on $z=0$.

Imposition of conditions (4.1) and (4.2) leads to the derivation of the pair of dual integral equations:

$$\int_0^\infty A_3p^2J_0(rp)\,\mathrm{d}p=0, \quad r<1 \tag{5.2}$$

$$\int_0^\infty A_3pJ_0(rp)\,\mathrm{d}p=f(r), \quad r>1. \tag{5.3}$$

To reduce the above equations to a single integral equation, we apply Sneddon's method [4] and put

$$A_3p=\int_1^\infty \phi_1(t)\sin(pt)\,\mathrm{d}t \tag{5.4}$$

where, for the convergence of the integral, we assume that

$$\lim_{t\to\infty}\phi_1(t)=0.$$

ollowing Das [3] $\phi(t)$ assumed in (5.4) can be found in the form

$$\phi_1(t) = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \frac{rf(r)}{\sqrt{r^2-t^2}}\,\mathrm{d}r, \quad 1 < t < \infty \tag{5.5}$$

The thermoelastic problem

ith $T(r,z)$ given by (5.1) solutions of (3.2) and (3.3) satisfying the regularity condition infinity are found in the form

$$\phi = \int_0^\infty (A_1 \exp(-n_2 z) + A_2 h_2 \exp(-n_1 z) + A_3 h_3 \exp(-nz))pJ_0(rp)\,\mathrm{d}p \tag{6.1}$$

$$\Omega = \int_0^\infty (A_2 \exp(-n_1 z) + A_3 h_1 \exp(-nz))pJ_0(rp)\,\mathrm{d}p. \tag{6.2}$$

he stress components σ_{zz}, σ_{rz} and the displacement component are

$$\sigma_{zz}(r,z) = -\int_0^\infty (A_1 \exp(-n_2 z) + A_2 h_2 \exp(-n_1 z) + A_3 h_3 \exp(-nz))p^3 J_0(rp)\,\mathrm{d}p \tag{6.3}$$

$$\sigma_{rz}(r,z) = -\int_0^\infty (A_1 n_2 \exp(-n_2 z) + A_2 h_2 n_1 \exp(-n_1 z) + A_3 h_3 n \exp(-nz))p^2 J_1(rp)\,\mathrm{d}p \tag{6.4}$$

nd

$$u_z(r,z) = -{}_1S_2 \int_0^\infty [-K_{12}^2 n_2 \exp(-n_2 z)A_1 + A_2 n_1 \exp(-n_1 z)(1 - K_{12}^2 h_2) + A_3 n \exp(-nz)(h_1 - K_{12}^2 h_3)pJ_0(rp)]\,\mathrm{d}p \tag{6.5}$$

ondition (4.3) gives

$$A_1 n_2 + A_2 h_2 n_1 + A_3 h_3 n = 0. \tag{6.6}$$

nposition of conditions (4.4) and (4.5) yields the pair of dual integral equations,

$$\int_0^\infty A_2 K_{21}^{-1} p^2 J_0(rp)\,\mathrm{d}p = -\int_0^\infty A_3 \frac{M}{L} J_0(rp)\,\mathrm{d}p, \quad r < 1 \tag{6.7}$$

$$\int_0^\infty A_2 h_2 \left(1 - \frac{K_{22}}{K_{21}}\right) p^3 J_0(rp)\,\mathrm{d}p = \int_0^\infty A_3 N \left(\frac{K_{22}}{L} - 1\right) pJ_0(rp)\,\mathrm{d}p, \quad r > 1. \tag{6.8}$$

Assuming

$$B(p)=h_2\left(1-\frac{K_{22}}{K_{21}}\right)p^3J_0(rp)A_2 \; -N\left(\frac{K_{22}}{L}-1\right)pJ_0(rp)A_3(p) \tag{6.9}$$

the above equations reduce to

$$\int_0^\infty p^{-1}B(p)J_0(rp)\,\mathrm{d}p=-\int_0^\infty\left[N\left(\frac{K_{22}}{L}-1\right)\right.$$

$$\left.+h_2(K_{21}-K_{22})\right]A_3J_0(rp)\,\mathrm{d}p,\quad r<1 \tag{6.10}$$

$$\int_0^\infty B(p)J_0(rp)\,\mathrm{d}p=0,\quad r>1 \tag{6.11}$$

7. Solution for a special case or temperature distribution

Let the annular region $z=0$, $1<r<a_1$, of each of the crack faces be maintained at a constant temperature V_0 below the temperature of the reference state. Then we have,

$$f(r)=-V_0H(a_1-r),\quad a_1>1 \tag{7.1}$$

where $H(r)$ is the Heavyside step function.

From (5.5) we get

$$\phi_1(t)=-\frac{2V_0}{\pi}\frac{tH(a_1-t)}{(a_1^2-t^2)^{1/2}} \tag{7.2}$$

Hence from (5.4) and (7.2) we get

$$A_3(p)=-\frac{2V_0}{\pi}\int_1^\infty\frac{tH(a_1-t)}{(a_1^2-t^2)^{1/2}}\frac{\sin pt}{p}\,\mathrm{d}t. \tag{7.3}$$

Making use of (5.1) and (7.3) we derive that

$$T(r,0)=-\frac{2V_0}{\pi}\sin^{-1}\left(\frac{a_1^2-1}{a_1^2-r^2}\right)^{1/2},\quad r<1 \tag{7.4}$$

Substituting (7.3), eqs (6.10), (6.11) take the form

$$\int_0^\infty Bp^{-1}J_0(rp)\,\mathrm{d}p$$

$$=V_0\left\{N\left(\frac{K_{22}}{L}-1\right)+h_2(K_{21}-K_{22})\frac{M}{L}\right\}(a_1^2-1)^{1/2},\quad r<1 \tag{7.5}$$

$$\int_0^\infty BJ_0(rp)\,\mathrm{d}p=0,\quad r>1 \tag{7.6}$$

From the following known results

$$\int_0^\infty J_0(rp)\sin p\,\mathrm{d}p = \begin{cases} \dfrac{1}{(1-r^2)^{1/2}} & \text{where } r<1 \\ 0 & \text{where } r>1 \end{cases}$$

and

$$\int_0^\infty J_0(rp)\frac{\sin p}{p}\,\mathrm{d}p = \begin{cases} \dfrac{\pi}{2}, & r<1 \\ \sin^{-1}\dfrac{1}{r}, & r>1 \end{cases}$$

it is clear that solution of the pair of dual integral eqs (7.5) and (7.6) is

$$B(p) = \frac{2}{\pi}V_0\left\{N\left(\frac{K_{22}}{L}-1\right)+h_2(K_{21}-K_{22})\frac{M}{L}\right\}(a_1^2-1)^{1/2}\sin p \tag{7.7}$$

From (6.9) and (7.7) we get

$$A_2 p = \frac{1}{h_2\left(1-\dfrac{K_{22}}{K_{21}}\right)}\left[\frac{2}{\pi}V_0\left\{N\left(\frac{K_{22}}{L}-1\right)\right.\right.$$
$$\left.\left. + h_2(K_{21}-K_{22})\frac{M}{L}\right\}(a_1^2-1)^{1/2}\frac{\sin p}{p^3} + A_3 N\left(\frac{K_{22}}{L}-1\right)\frac{1}{p^2}\right] \tag{7.8}$$

Making use of (7.3), (7.8), (6.6), (6.3) and (6.5) we derive that

$$\sigma_{zz}(r,0) = \frac{2V_0}{\pi}\left\{N\left(1-\frac{K_{22}}{L}\right)+h_2(K_{22}-K_{21})\frac{M}{L}\right\}\left(\frac{a_1^2-1}{1-r^2}\right)^{1/2}, \quad r<1$$

$u_z(r,0)$

$$= \begin{cases} \dfrac{2V_0}{\pi}\,{}_1S_2\left\{\dfrac{N(K_{22}-L)}{Lh_2(K_{21}-K_{22})}+\dfrac{M}{L}\right\}a_1\left[E\left(\dfrac{r}{a_1}\right)-E\left(\dfrac{r}{a_1},\sin^{-1}\dfrac{1}{r}\right)\right], & 1<r\leqslant a_1 \\ \dfrac{2V_0}{\pi}\,{}_1S_2\left\{\dfrac{N(K_{22}-L)}{Lh_2(K_{21}-K_{22})}+\dfrac{M}{L}\right\}\left[r\left\{E\left(\dfrac{a_i}{r}\right)-E\left(\dfrac{a_1}{r},\sin^{-1}\dfrac{1}{a_1}\right)\right\}\right. & \\ \quad \left.-\dfrac{r^2-a_1^2}{r}\left\{K\left(\dfrac{a_1}{r}\right)-F\left(\dfrac{a_1}{r},\sin^{-1}\dfrac{1}{a_1}\right)\right\}\right], & r\geqslant a_1 \end{cases}$$

The stress intensity factor

$$\lim_{r\to 1-}(1-r)^{1/2}\sigma_{zz}(r,0)$$
$$= \frac{\sqrt{2}V_0}{\pi}\left\{N\left(1-\frac{K_{22}}{L}\right)+h_2(K_{22}-K_{21})\frac{M}{L}\right\}(a_1^2-1)^{1/2}, \quad a_1>1.$$

Table 1. Variation of $\sigma_{zz}(r, 0)$ and $\sigma'_{zz}(r, 0)$.

r	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\frac{\sigma_{zz}}{V_0} \times 10^{-6}$	9·54701	9·69504	9·95782	10·36443	10·96868	11·87395	13·30148	15·83193	21·79257
$\frac{\sigma'_{zz}}{V_0} \times 10^{-6}$	9·25133	9·39477	9·64942	10·04343	10·62897	11·5062	12·88952	15·34160	21·11763

Table 2. Variation of $u_z(r, 0)$ and $u'_z(r, 0)$.

r	1·1	1·2	1·3	1·5	1·6	1·8	1·9	2	2·5	3	3·5	4	5
$\frac{u_z}{V_0} \times 10^{6}$	15·90051	20·95862	24·46590	27·46776	27·64312	26·06231	24·40413	21·45526	13·16930	9·734483	8·12078	7·06641	5·0503
$\frac{u'_z}{V_0} \times 10^{6}$	17·07227	22·50312	26·26885	29·49193	29·68022	27·98291	26·20253	23·0364	14·13980	10·45223	8·71923	7·58716	5·42247

7. Numerical results

We consider magnesium which is transversely isotropic and for which the roots of (3.4) and (3.11) are real.

The elastic constants s_{ij} for this material expressed in $10^{-12}\,\mathrm{cm}^2/\mathrm{dyn}$ are

$$S_{11} = 2{\cdot}23 \qquad S_{33} = 1{\cdot}9 \qquad S_{44} = 5{\cdot}9$$

$$S_{12} = -0{\cdot}77 \quad S_{13} = -0{\cdot}45$$

The constant L has been taken equal to unity. For polycrystalline magnesium which is isotropic we put in Das's [3] results

Coefficient of linear thermal expansion	$\alpha_t = 27{\cdot}15 \times 10^{-6}\,\mathrm{cm}/{}^\circ\mathrm{C\,cm}$
Young's modulus	$E = 0{\cdot}41 \times 10^{12}\,\mathrm{dyn/cm}^2$
Poisson's ratio	$\nu = \frac{1}{3}$

and denote the corresponding normal components of stress and displacement by σ'_{zz} and u'_z respectively.

The results are given in tables 1 and 2 and figures 2 and 3.

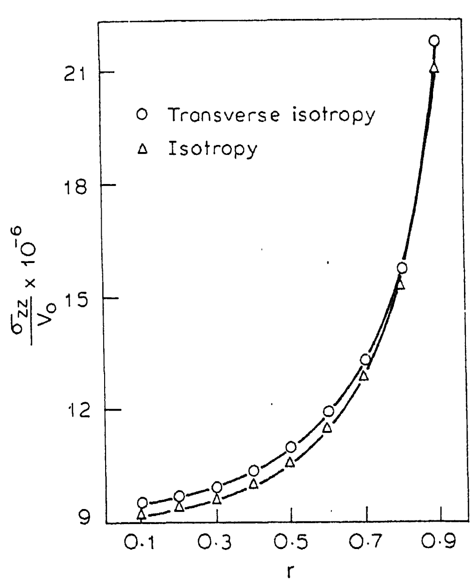

Figure 2.

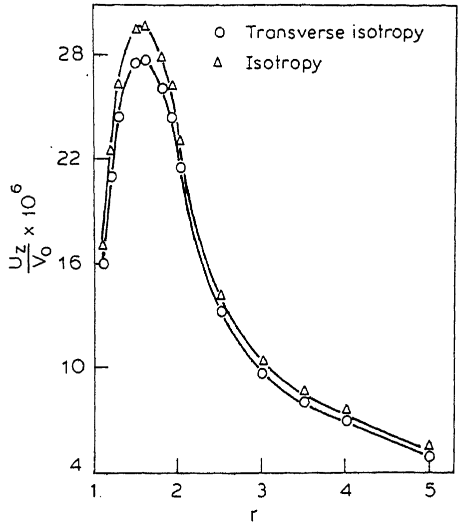

Figure 3.

8. Concluding remarks

From tables 1 and 2, and figures 2 and 3, it is clear that, as expected, the normal component of stress σ_{zz} acting on the crack plane $z = 0$ has an infinite discontinuity at points on the circle $r = 1$, the exterior of which is the crack. It is further seen that in magnesium which is transversely isotropic and polycrystalline magnesium which is isotropic, the differences in the values of both $\sigma_{zz}(r, 0)$, $r < 1$ and the normal displacement $u_z(r, 0)$, $r > 1$ are not very marked. Moreover, in both the cases $u_z(r, 0)$, $r > 1$ first increases rapidly with r, attains a maximum and then diminishes steadily till it vanishes for large values of r.

References

[1] Elliot H A, Three-dimensional stress distributions in hexagonal aerolotropic crystals, *Proc. Cambridge Philos Soc.* **44** (1948), 522–533

[2] Singh Avtar, Axisymmetrical thermal stresses in transversely isotropic bodies, *Arch. Mech. Stosow*, **3** (1960)·12

[3] Das B R, Some axially symmetric thermal stress distributions in elastic solids containing cracks-1. An external crack in an infinite solid. *Int. J. Eng. Sci.* **9** (1971) pp. 469–478 (Pergamon Press)

[4] Sneddon I N, The elementary solution of dual integral equations, *Proc. Glasgow. Math. Assoc.*, **4** (1960) 108

Authors' note on the paper "Rheology of polarizable non-piezoelectromagnetic material in relativity"

With deep regret we state that a substantial portion of our paper [2], named in the title, had appeared in the paper [1] coauthored by one of us (M A Shah) with L Radhakrishna. We realise that repeating much of the material was uncalled for and that at any rate a reference should have been made to [1]. Certain unfortunate circumstances relating to the earlier collaborative work led to the lapse. We offer our sincere apologies.

References

[1] L Radhakrishna and Madhubala A Shah, Rheometrodynamics, *Proceedings of the Workshop on Solid Mechanics*, March 13 to 16, 1985, Department of Mathematics, University of Roorkee, Roorkee

[2] M A Shah and B B Waghmode, Rheology of polarizable non-piezoelectromagnetic material in relativity, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **96** (1987) 103–118

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The above note arose out of our following up a complaint from L Radhakrishna. Repetition of published material, except when clearly meant to be for expository purposes, and any suppression of references is viewed by the Editors with serious concern.

—Editor

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